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THE BINARY STARS

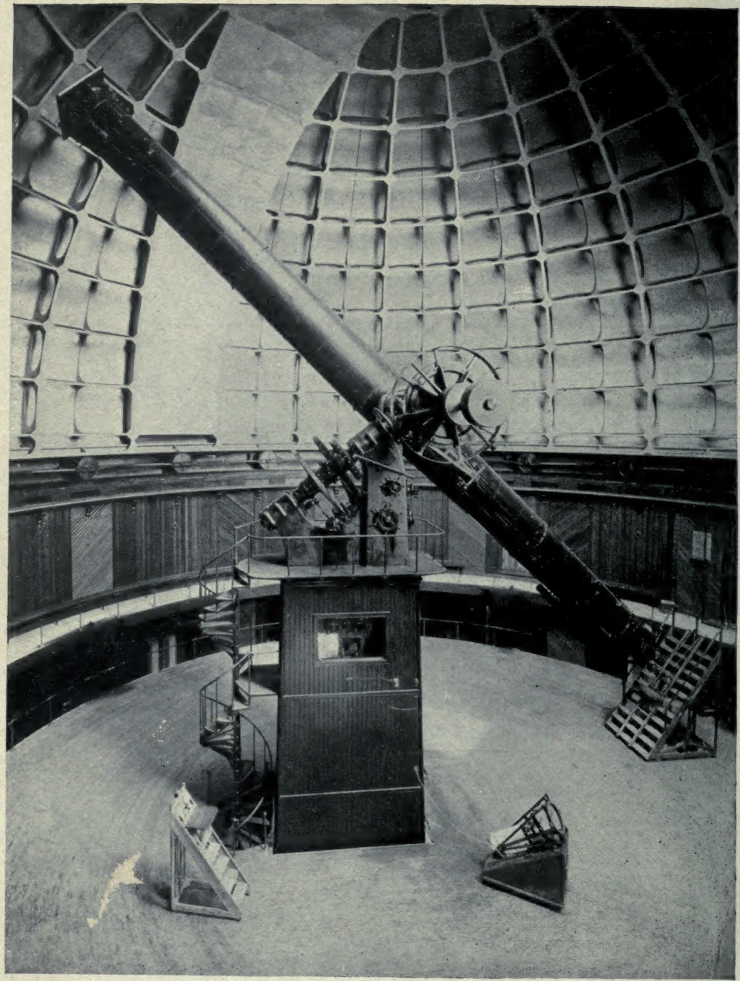

Plate I. The Thirty-Six-Inch Refractor of the Lick Observatory

THE BINARY STARS

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DOUGLAS C. McMURTRIE

TO

SHERBURNE WESLEY BURNHAM

THIS VOLUME

IS GRATEFULLY INSCRIBED

PREFACE

Credit has been given on many pages of this volume for assistance received in the course of its preparation; but I desire to express in this more formal manner my special gratitude, first of all, to my colleague, Dr. J. H. Moore, for contributing the valuable chapter on The Radial Velocity of a Star; also to Director E. C. Pickering and Miss Annie J. Cannon, of the Harvard College Observatory, for generously permitting me to utilize data from the New Draper Catalogue of Stellar Spectra; to Professor H. N. Russell, of Princeton University, Professor F. R. Moulton, of the University of Chicago, Professor E. E. Barnard, of the Yerkes Observatory, and Dr. H. D. Curtis, my colleague, for putting at my disposal published and unpublished material; and, finally, to Director W. W. Campbell, for his constant interest in and encouragement of my work. Nearly all of the manuscript has been read by Dr. Campbell and by Dr. Curtis, and several of the chapters also by Dr. Moore and by Dr. Reynold K. Young, and I am deeply indebted to them for their friendly criticism.

The book appears at this particular time in order that it may be included in the series of Semi-Centennial Publications issued by the University of California.

R. G. Aitken

December, 1917

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INTRODUCTION

It is the object of this volume to give a general account of our present knowledge of the binary stars, including such an exposition of the best observing methods and of approved methods of orbit computation as may make it a useful guide for those who wish to undertake the investigation of these systems; and to present some conclusions based upon the author's own researches during the past twenty years.

The term *binary star* was first used by Sir William Herschel, in 1802, in his paper "On the Construction of the Universe," to designate "a real double star—the union of two stars, that are formed together in one system, by the laws of attraction."

The term *double star* is of earlier origin; its Greek equivalent was, in fact, used by Ptolemy to describe the appearance of ν *Sagittarii*, two fifth magnitude stars whose angular separation is about 14′, or a little less than half of the Moon's apparent diameter. It is still occasionally applied to this and other pairs of stars visible to the unaided eye, but is generally employed to designate pairs separated by only a few seconds of arc and therefore visible as two stars only with the aid of a telescope.

Not every double star is a binary system, for, since all of the stars are apparently mere points of light projected upon the surface of the celestial sphere, two unrelated stars may appear to be closely associated simply as the result of the laws of perspective. Herschel draws the distinction between the two classes of objects in the following words:

" . . . if a certain star should be situated at any, perhaps immense, distance behind another, and but little deviating from the line in which we see the first, we should have the appearance of a double star. But these stars being totally unconnected would not form a binary system. If, on the contrary, two stars should really be situated very near each other, and at the same time so far insulated as not to be materially affected by neighboring stars, they will then compose a separate system, and remain united by the bond of their mutual gravitation toward each other. This should be called a real double star."

Within the last thirty years we have become acquainted with a class of binary systems which are not double stars in the ordinary sense of the term at all, for the two component stars are not separately visible in any telescope. These are the *spectroscopic binary stars*, so named because their existence is demonstrated by a slight periodic shifting to and fro of the lines in their spectra, which, as will be shown, is evidence of a periodic variation in the radial velocity (the velocity in the line of sight, toward or away from the observer) of the star. The only differences between the spectroscopic and the visual binary ("real double") stars are those which depend upon the degree of separation of the two components. The components of a spectroscopic binary, are, in general, less widely separated than those of a visual binary, consequently they are not separately visible even with the most powerful telescopes and the systems have relatively short periods of revolution.

In the present volume the two classes will be regarded as members of a single species.

CHAPTER I

HISTORICAL SKETCH: THE EARLY PERIOD

The first double star was discovered about the year 1650 by the Italian astronomer, Jean Baptiste Riccioli. This was ζ *Ursae Majoris* (*Mizar*). It is a remarkable coincidence that *Mizar* was also the first double star to be observed photographically, measurable images being secured by G. P. Bond, at the Harvard College Observatory in 1857; and that its principal component was the first spectroscopic binary to be discovered, the announcement being made by E. C. Pickering in 1889.

In 1656, Huyghens saw θ *Orionis* resolved into the three principal stars of the group which form the familiar Trapezium, and, in 1664, Hooke noted that γ *Arietis* consisted of two stars. At least two additional pairs, one of which proved to be of more than ordinary interest to astronomers, were discovered before the close of the Seventeenth Century. It is worthy of passing note that these were southern stars, not visible from European latitudes,—α *Crucis*, discovered by the Jesuit missionary, Father Fontenay, at the Cape of Good Hope, in 1685, and α *Centauri*, discovered by his confrère, Father Richaud, while observing a comet at Pondicherry, India, in December, 1689.

These discoveries were all accidental, made in the course of observations taken for other purposes. This is true also of the double stars found in the first three-quarters of the Eighteenth Century. Among these were the discoveries of γ *Virginis*, in 1718, and of *Castor*, in 1719, by Bradley and Pound, and of 61 *Cygni*, by Bradley, in 1753.

No suspicion seems to have been entertained by these astronomers or by their contemporaries that the juxtaposition of the two star images in such pairs was other than optical, due to the chance positions of the Earth and the two stars in nearly a straight line. They were therefore regarded as mere curiosities, and no effort was made to increase their number; nor were observations of the relative positions of the two components recorded except in descriptive terms. Father Feuille, for instance, on July 4, 1709, noted that the fainter star in the double, α *Centauri*, "is the more western and their distance is equal to the diameter of this star," and Bradley and Pound entered in their observing book, on March 30, 1719, that "the direction of the double star α of *Gemini* was so nearly parallel to a line through κ and σ of *Gemini* that, after many trials, we could scarce determine on which side of σ the line from κ parallel to the line of their direction tended; if on either, it was towards β."

Halley's discovery, in 1718, that some of the brighter stars, *Sirius*, *Arcturus*, *Aldebaran*, were in motion, having unmistakably changed their positions in the sky since the time of Ptolemy, unquestionably stimulated the interest of astronomers in precise observations of the stars. These researches and their results, in turn, were probably largely responsible for the philosophical speculations which began to appear shortly after the middle of the Eighteenth Century as to the possibility of the existence of systems among the stars. Famous among the latter are the *Cosmologische Briefe*,[1] published in 1761 by Lambert, in which it is maintained that the stars are suns and are accompanied by retinues of planets. Lambert, however, apparently did not connect his speculations with the double stars then known. Six years later, in 1767, John Michell, in a paper read before the Royal Society of London, presented a strong argument, based upon the theory of probabilities, that "such double stars, etc., as appear to consist of two or more stars placed near together, do really consist of stars placed near together, and under the influence of some general law, whenever the probability is very great, that there would not have been any such stars so near together, if all those that are not less bright than themselves had been scattered at random through the whole heavens." Michell thus has the credit of being the first to establish the probability of

[1] Cosmologische Briefe über die Einrichtung des Weltbaues, Ausgefertigt von J. H. Lambert, Augsburg, 1761.

the existence of physical systems among the stars; but there were no observational data to support his deductions and they had no direct influence upon the progress of astronomy.

The real beginning of double star astronomy dates from the activities of Christian Mayer and, in particular, of Sir William Herschel, in the last quarter of the Eighteenth Century. If a definite date is desired we may well follow Lewis in adopting the year 1779, for that year is marked by the appearance of Mayer's small book entitled "De novis in Coelo Sidereo Phaenominis in miris Stellarum fixarum Comitibus," wherein he speculates upon the possibility of small suns revolving around larger ones, and by the beginning of Herschel's systematic search for double stars.

The difference between Mayer's speculations and earlier ones is that his rest in some degree at least upon observations. These were made with an eight-foot Bird mural quadrant at Mannheim, in 1777 and 1778. At any rate, in his book just referred to, he publishes a long list of faint companions observed in the neighborhood of brighter stars.[2] As one result of his observations he sent to Bode, at Berlin, the first collection or catalogue of double stars ever published. The list contained earlier discoveries as well as his own and is printed in the *Astronomisches Jahrbuch* for the year 1784 (issued in 1781) under the caption, "Verzeichnis aller bisher entdeckten Doppelsterne." The following tabulation gives the first five entries:

Grösse	Gerade Aufst. G. M.		Abweichung G. M.	Unterschied in der Aufst. Sec.	Unterschied in der Abw. Sec.	Abstand Sec.	Stellung des Kleinern
Andromeda beyde 9ter	8	38	29 45 N	45	24	46	S. W.
Andromeda beyde 9ter	13	13	20 18 N	15	29	32	S. O.
ζ Fische 6. und 7ter	15	33	6 25 N	22	9	24	N. O.
bey μ Fische beyde 7ter	19	24	5 0 N	0	4	4	S.
γ Widder beyde 5ter	25	22	18 13 N	3	12	12	S. W.

[2] This list, rearranged according to constellations, was reprinted by Schjellerup in the journal *Copernicus*, vol. 3, p. 57, 1884.

In all, there are eighty entries, many of which, like *Castor* and γ *Virginis*, are among the best known double stars. Others are too wide to be found even in Herschel's catalogues and a few cannot be identified with certainty. Southern pairs, like α *Centauri*, are of course not included, and curiously enough, θ *Orionis* is not listed. The relative positions given for the stars in each pair are little better than estimates, for precise measures were not practicable until the invention of the 'revolving micrometer'.

In his comments on Mayer's catalogue Bode points out that careful observations of such pairs might become of special value in the course of time for the discovery of proper motions, since it would be possible to recognize the fact of motion in one or the other star as soon as the distance between them had changed by a very few seconds of arc. Mayer himself seems to have had proper motions in view in making his observations and catalogue rather than any idea of orbital motions.

Sir William Herschel "began to look at the planets and the stars" in May, 1773; on March 1, 1774, "he commenced his astronomical journal by noting that he had viewed Saturn's ring with a power of forty, appearing 'like two slender arms' and also 'the lucid spot in Orion's sword belt'." The earliest double star measure recorded in his first catalogue is that of θ *Orionis*, on November 11, 1776, and he made a few others in the two years following. It was not until 1779, however, that he set to work in earnest to search for these objects, for it was then that he conceived the idea of utilizing them to test a method of measuring stellar parallax suggested long before by Galileo. The principle involved is very simple. If two stars are in the same general direction from us and one is comparatively near us while the other is extremely distant, the annual revolution of the Earth about the Sun will produce a periodic variation in the relative positions of the two. As a first approximation, we may regard the more distant star as absolutely fixed and derive the parallax of the nearer one from the measured displacements.

It seemed clear to Herschel that the objects best fitted for such an investigation were close double stars with components of unequal brightness. He pointed out in his paper "On the Parallaxes of the Fixed Stars", read before the Royal Society in 1781, that the displacement could be more easily and certainly detected in a close double star than in a pair of stars more widely separated and also that in the former case the observations would be free from many errors necessarily affecting the measures in the latter.

"As soon as I was fully satisfied," he continues, "that in the investigation of parallax the method of double stars would have many advantages above any other, it became necessary to look out for proper stars. This introduced a new series of observations. I resolved to examine every star in the heavens with the utmost attention and a very high power, that I might collect such materials for this research as would enable me to fix my observations upon those that would best answer my ends."

In this reasoning, Herschel assumes that there is no physical connection between the components of such close double stars, —a fact upon which every writer on the history of double star astronomy has commented. This was not an oversight on his part, for at the close of his first catalogue of double stars he remarks, "I preferred that expression (*i.e.*, double stars) to any other, such as Comes, Companion, or Satellite; because, in my opinion, it is much too soon to form any theories about small stars revolving round large ones, and I therefore thought it advisable carefully to avoid any expression that might convey that idea."

Herschel's telescopes were more powerful than any earlier ones and with them he soon discovered a far larger number of double stars than he had anticipated. With characteristic thoroughness he nevertheless decided to carry out his plan of examining "every star in the heavens," and carefully recorded full details of all his observations. These included a general description of each pair and also estimates, or measures with the "revolving micrometer," or "lamp micrometer," both invented by himself, of the apparent distance between the two components and of the direction of the smaller star from the larger. The direction, or position angle, of the smaller star, by his definition, was the angle at the larger star between the line joining the two stars and a line parallel to the celestial equator. The angle was always made less than 90°, the letters, *nf*, *sf*, *sp*, and *np* being added to designate the quadrant. His first catalogue, presented to the Royal Society on January 10, 1782, contains 269 double stars, "227 of which, to my present knowledge, have not been noticed by any person." A second catalogue, containing 434 additional objects, was presented to the same society in 1784. The stars in these catalogues were divided into six classes according to angular separation.

"In the first," he writes, "I have placed all those which require indeed a very superior telescope, the utmost clearness of air, and every other favorable circumstance to be seen at all, or well enough to judge of them. . . . In the second class I have put all those that are proper for estimations by the eye or very delicate measures of the micrometer. . . . In the third class I have placed all those . . . that are more than five but less than 15″ asunder; . . . The fourth, fifth, and sixth classes contain double stars that are from 15″ to 30″, from 30″ to 1′ and from 1′ to 2′ or more asunder."

Class I, in the two catalogues, includes ninety-seven pairs, and contains such systems as *τ Ophiuchi*, *δ Herculis*, *ε Boötis*, *ξ Ursae Majoris*, 4 *Aquarii*, and *ζ Cancri*. In general, Herschel did not attempt micrometer measures of the distances of these pairs because the finest threads available for use in his micrometers subtended an angle of more than 1″. The following extracts will show his method of estimating the distance in such cases and of recording the position angle, and also the care with which he described the appearance of each object. The dates of discovery, or of the first observation, here printed above the descriptions, are set in the margin at the left in the original.

H. I. September 9, 1779

ε Boötis, Flamst. 36. Ad dextrum femur in perizomate. Double. Very unequal. *L.* reddish; *S.* blue, or rather a faint lilac. A very beautiful object. The vacancy or black division between them, with 227 is ¾ diameter of

S.; with 460, 1¼ diameter of *L.;* with 932, near 2 diameters of *L.;* with 1,159, still farther; with 2,010 (extremely distinct), ⅔ diameters of *L.* These quantities are a mean of two years' observation. Position 31° 34′ n preceding.

H. 2. May 2, 1780

ξ Ursae Majoris. Fl. 53. In dextro posteriore pede. Double. A little unequal. Both w [white] and very bright. The interval with 222 is ⅔ diameter of *L.;* with 227, 1 diameter of *L;* with 278, near 1½ diameter of *L.* Position 53° 47′ s following.

Careful examination of the later history of the stars of Herschel's Class I shows that the majority had at discovery an angular separation of from 2″ to 3.5″; a half dozen pairs as wide as 5″ are included (one with the ms. remark, "Too far asunder for one of the first class"); and a number as close as or closer than 1″. Seven of these stars do not appear in the great catalogue of Struve, but five of these have been recovered by later observers, leaving only two that cannot be identified.

In passing judgment upon the accuracy, or the lack of it, in Herschel's measures of double stars, it is necessary to hold in mind the conditions under which he had to work. His reflectors (all of his own construction) were indeed far more powerful telescopes than any earlier ones, especially the "twenty-feet reflector," with mirror of eighteen and three-quarter inches aperture, and the great "forty-feet telescope," with its four-foot mirror. But these telescopes were unprovided with clock-work; in fact their mountings were of the alt-azimuth type. It was therefore necessary to move the telescope continuously in both coordinates to keep a star in the field of view and the correcting motions had to be particularly delicate when high-power eye-pieces, such as are necessary in the observation of close double stars, were employed. Add the crude forms of micrometers at his disposal, and it will appear that only an observer of extraordinary skill would be able to make measures of any value whatever.

No further catalogues of double stars were published by Herschel until June 8, 1821, about a year before his death, when he presented to the newly founded Royal Astronomical

Society a final list of 145 new pairs, not arranged in classes, and, for the most part, without measures.

After completing his second catalogue, in 1784, Herschel seems to have given relatively little attention to double stars until about the close of the century and, though he doubtless tested it fully, there is no mention of his parallax method in his published writings after the first paper on the subject. A thorough review of his double star discoveries which he instituted about the year 1797 with careful measures, repeated in some cases on many nights in different years, revealed a remarkable change in the relative positions of the components in a number of double stars during the interval of nearly twenty years since their discovery, but this change was of such a character that it could not be produced by parallax.

We have seen that, in 1782, Herschel considered the time not ripe for theorizing as to the possible revolution of small stars about larger ones. Probably no astronomer of his own or of any other age was endowed in a higher degree than Herschel with what has been termed the scientific imagination; certainly no one ever more boldly speculated upon the deepest problems of sidereal astronomy; but his speculations were the very opposite of guesswork, invariably they were the results of critical analyses of the data given by observation and were tested by further observations when possible. Michell, in 1783, applied his earlier argument from the theory of probabilities to the double stars in Herschel's first catalogue and concluded that practically all of them were physical systems; but it was not until July, 1802, that Herschel himself gave any intimation of holding similar views. On that date he presented to the Royal Society a paper entitled "Catalogue of 500 new Nebulae, nebulous Stars, planetary Nebulae, and Clusters of Stars; with Remarks on the Construction of the Heavens", in which he enumerates "the parts that enter into the construction of the heavens" under twelve heads, the second being, "II. Of Binary sidereal Systems, or double Stars." In this section he gives the distinction between optical and binary systems quoted in my Introduction and argues as to the possibility of systems of the latter type under the law of gravitation.

On June 9, 1803, followed the great paper in which he gave the actual demonstration, on the basis of his measures, that certain double stars are true binary systems. This paper, the fundamental document in the physical theory of double stars, is entitled, "Account of the Changes that have happened, during the last Twenty-five Years, in the relative Situation of Double-stars; with an Investigation of the Cause to which they are owing." After pointing out that the actual existence of binary systems is not proved by the demonstration that such systems *may* exist, Herschel continues, "I shall therefore now proceed to give an account of a series of observations on double stars, comprehending a period of about twenty-five years which, if I am not mistaken, will go to prove, that many of them are not merely double in appearance, but must be allowed to be real binary combinations of two stars, intimately held together by the bonds of mutual attraction."

Taking *Castor* as his first example, he shows that the change in the position of the components is real and not due to any error of observation. Then, by a masterly analysis of every possible combination of motions of the Sun and the components in this, and in five other systems, he proves that orbital motion is the simplest and most probable explanation in any one case, and the *only reasonable one* when all six are considered. His argument is convincing, his conclusion incontrovertible, and his paper, a year later, containing a list of fifty additional double stars, many of which had shown motion of a similar character, simply emphasizes it.

This practically concluded Sir William Herschel's contributions to double star astronomy, for his list of 145 new pairs, published in 1821, was based almost entirely upon observations made before 1802. In fact, little was done in this field by any one from 1804 until about 1816. Sir John Herschel, in that year, decided to review and extend his father's work and had made some progress when Sir James South, who had independently formed similar plans, suggested that they cooperate.

The suggestion was adopted and the result was a catalogue of 380 stars, based upon observations made in the years 1821 to 1823 with South's five-foot and seven-foot refractors, of $3\frac{3}{4}''$ and $5''$ aperture respectively. These telescopes were mounted equatorially but were not provided with driving-clocks. They were, however, equipped with micrometers in which the parallel threads were fine spider lines. The value of the catalogue was greatly increased by the inclusion of all of Sir William Herschel's measures, many of which had not before been published.

Both of these astronomers devoted much attention to double stars in following years, working separately however, South with his refractors, Herschel with a twenty-foot reflector (eighteen-inch mirror) and later with the five-inch refractor which he had purchased from South. They not only remeasured practically all of Sir William Herschel's double stars, some of them on many nights in different years, but they, and in particular Sir John Herschel, added a large number of new pairs. Indeed, so numerous were J. Herschel's discoveries and so faint were many of the stars that he deemed some apology necessary. He says, " . . . so long as no presumption *a priori* can be adduced why the most minute star in the heavens should not give us that very information respecting parallax, proper motion, and an infinity of other interesting points, which we are in search of, and yet may never obtain from its brighter rivals, the minuteness of an object is no reason for neglecting its examination. . . . But if small double stars are to be watched, it is first necessary that they should become known; nor need we fear that the list will become overwhelming. It will be curtailed at one end, by the rejection of uninteresting and uninstructive objects, at least as fast as it is increased on the other by new candidates." The prediction made in the closing sentence has not been verified; on the contrary, the tendency today is rather to include in the great reference catalogues every star ever called double, even those rejected later by their discoverers.

The long series of measures and of discoveries of double stars by Herschel and South were of great value in themselves and perhaps of even greater value in the stimulus they gave to the observation of these objects by astronomers generally, and well merited the gold medals awarded to their authors by the Royal Astronomical Society. The measures, however, are now assigned small weight on account of the relatively large errors of observation due to the conditions under which they were of necessity made; and of the thousands of new pairs very few indeed have as yet proved of interest as binaries. The great majority are too wide to give the slightest evidence of orbital motion in the course of a century.

The true successor to Sir William Herschel, the man who made the next real advance in double star astronomy, an advance so great that it may indeed be said to introduce a new period in its history, was F. G. W. Struve. Wilhelm Struve became the director of the observatory at Dorpat, Russia, in 1813, and soon afterwards began measuring the differences in right ascension and in declination between the components of double stars with his transit instrument, the only instrument available. A little later he acquired a small equatorial, inferior to South's, with which he continued his work, and, in 1822, he published his "Catalogus 795 stellarum duplicium." This volume is interesting but calls for no special comment because Struve's great work did not really begin until two years later, in November, 1824, when he received the celebrated Fraunhofer refractor.

This telescope as an instrument for precise measurements was far superior to any previously constructed. The tube was thirteen feet long, the objective had an aperture of nine Paris inches,[3] the mounting was equatorial and of very convenient form, and, best of all, was equipped with an excellent driving clock. So far as I am aware, this was the first telescope employed in actual research to be provided with clock-work though Passement, in 1757, had "presented a telescope to the King [of France], so accurately driven by clock-work that it would follow a star all night long." A finder of two and one-half inches aperture and thirty inches focus, a full battery of

[3] This is Struve's own statement. Values ranging from 9½ to 9.9 inches (probably English inches) are given by different authorities.

eye-pieces, and accurate and convenient micrometers completed the equipment, over which Struve was pardonably enthusiastic. After careful tests he concluded that "we may perhaps rank this enormous instrument with the most celebrated of all reflectors, *viz.*, Herschel's."

Within four days after its arrival Struve had succeeded in erecting it in a temporary shelter and at once began the first part of his well-considered program of work. His object was the study of double stars as physical systems and so carefully had he considered all the requirements for such an investigation and so thorough, systematic, and skilful was the execution of his plans that his work has served as a model to all of his successors. His program had three divisions: the search for double stars; the accurate determination of their positions in the sky with the meridian circle as a basis for future investigations of their proper motions; and the measurement with the micrometer attached to the great telescope of the relative positions of the components of each pair to provide the basis for the study of motions within the system.

The results are embodied in three great volumes, familiarly known to astronomers as the 'Catalogus Novus', the 'Positiones Mediae', and the 'Mensurae Micrometricae'. The first contains the list of the double stars found in Struve's survey of the sky from the North Pole to $-15°$ declination. For the purposes of this survey he divided the sky into zones from $7\frac{1}{2}°$ to $10°$ wide in declination and swept across each zone from north to south, examining with the main telescope all stars which were bright enough, in his estimation, to be visible in the finder at a distance of $20°$ from the full Moon. He considered that these would include all stars of the eighth magnitude and the brighter ones of those between magnitudes eight and nine. Struve states that the telescope was so easy to manipulate and so excellent in its optical properties that he was able to examine 400 stars an hour; and he did, in fact, complete his survey, estimated to embrace the examination of 120,000 stars, in 129 nights of actual work in the period from November, 1824, to February, 1827.

Since each star had to be chosen in the finder, then brought into the field of view of the large telescope, examined, and, if double, entered in the observing record, with a general description, and an approximate position determined by circle readings, it is obvious that at the rate of 400 stars an hour, only a very few seconds could be devoted to the actual examination of each star. If not seen double, or suspiciously elongated at the first glance, it must, as a rule, have been passed over. Struve indeed definitely states that at the first instant of observation it was generally possible to decide whether a star was single or double. This is in harmony with my own experience in similar work, but I have never been content to turn away from a star apparently single until satisfied that further examination on that occasion was useless. As a matter of fact, later researches have shown that Struve overlooked many pairs within his limits of magnitude and angular separation, and hence easily within the power of his telescope; but even so the *Catalogus Novus*, with its short supplement, contains 3,112 entries. In two instances a star is accidentally repeated with different numbers so that 3,110 separate systems are actually listed. Many of these had been seen by earlier observers and a few that had entirely escaped Struve's own search were included on the authority of Bessel or some other observer.

Struve did not stop to make micrometer measures of his discoveries while engaged in his survey, and the *Catalogus Novus* therefore gives simply a rough classification of the pairs according to their estimated angular separation, with estimates of magnitude and approximate positions in the sky based on the equatorial circle readings. He rejected Herschel's Classes V and VI, taking 32″ as his superior limit of distance and dividing the stars within this limit into four classes: (1) Those under 4″; (2) those between 4″ and 8″; (3) those between 8″ and 16″; and (4) those between 16″ and 32″. Stars in the first class were further distinguished as of three grades by the use of the adjectives *vicinae*, *pervicinae*, and *vicinissimae*. The following lines will illustrate the form of the catalogue, the numbers in the last column indicating the stars that had been published in his prior catalogue of 795 pairs:

Numerus	Nomen Stellae	A. R.	Decl.	Descriptio	Num. C. P.
1		oh 0.0′	+36° 15′	II (8.9) (9)	
2	Cephei 316	−0.0	+78 45	I (6.7) (6.7), *vicinae*	
3	Andromedae 31	−0.4	+45 25	II (7.8) (10) = H.II 83	1
4		−0.9	+ 7 29	II (9), Besseli mihi non inventa	
5	34 Piscium	−1.1	+10 10	III (6) (10), Etiam Besseli	

The *Catalogus Novus*, published in 1827, furnished the working program on which Struve's other two great volumes were based, though the *Positiones Mediae* includes meridian circle measures made as early as 1822, and the *Mensurae Micrometricae* some micrometer measures made in the years 1824 to 1828. Micrometer work was not actively pushed until 1828 and four-fifths of the 10,448 measures in the 'Mensurae' were made in the six years 1828–1833. The final measures for the volume were secured in 1835 and it was published in 1837. The meridian observations were not completed until 1843, and the *Positiones Mediae* appeared nine years later, in 1852.

The latter volume does not specially concern us here for it is essentially a star catalogue, giving the accurate positions of the Σ (the symbol always used to designate Struve's double stars) stars for the epoch 1830.0. The *Mensurae Micrometricae*, on the other hand, merits a more detailed description, for the measures within it hold in double star astronomy a position comparable to that of Bradley's meridian measures in our studies of stellar proper motions. They are fundamental. The book is monumental in form as well as in contents, measuring seventeen and one-half inches by eleven. It is, as Lewis remarks, not to be taken lightly, and its gravity is not lessened by the fact that the notes and the Introduction of 180 pages are written in Latin. Every serious student of double stars, however, should read this Introduction carefully.

Looking first at the actual measures, we find the stars arranged in eight classes, Class I of the *Catalogus Novus* being divided into three, to correspond to the grades previously defined by adjectives, and Classes III and IV, into two each. The upper limits of the eight classes, accordingly, are 1, 2, 4, 8, 12, 16, 24, and 32″, respectively. The stars in each class are further distinguished according to magnitude, being graded as *lucidae* if both components of the pair are brighter than 8.5 magnitude, and *reliquae* if either component is fainter than this.

Sir John Herschel had early proposed that the actual date of every double star measure be published and that it be given in years and the decimal of a year. About the year 1828 he further suggested that position angles be referred to the north pole instead of to the equator as origin and be counted through 360°. This avoids the liability to mistakes pertaining to Sir William Herschel's method. Both suggestions were adopted by Struve and have been followed by all later observers. Generally the date is recorded to three decimals, thus defining the day, but Struve gives only two. The position angle increases from North (0°) through East, or following (90°), South (180°), and West or preceding (270°).

The heading of the first section, and the first entry under it will illustrate the arrangement of the measures in the *Mensurae Micrometricae:*

DUPLICES LUCIDAE ORDINIS PRIMI

Quarum distantiae inter 0″.00 et 1″.00

Epocha	Amplif.	Distant.	Angulus	Magnitudines
2	Cephei 316.	$\alpha = 0^h 0.'0$.	$\delta = 78° 45'$	
Major—6.3 flava;		*minor =6.6 certe flavior*		
1828.22	600	0.72″	342.5°	6.5, 7
1828.27	600	0.84	343.4	6.5, 7
1832.20	600	0.94	339.3	6, 6
1832.24	480	0.70	337.5	6, 6.5
1833.34	800	0.85	344.8	6.5, 6.5*m*
Medium 1830.85		0.810	341.50	

The Introduction contains descriptions of the plan of work, the instrument, and the methods of observing, and thorough discussions of the observations. The systems of magnitudes and of color notation, the division of the stars into classes by distance and magnitude, the proper and orbital motions detected, are among the topics treated. One who does not care to read the Latin original will find an excellent short summary in English in Lewis's volume on the Struve Double Stars published in 1906 as Volume LVI of the *Memoirs of the Royal Astronomical Society of London.* Three or four of Struve's general conclusions are still of current interest and importance. He concludes, for example, that the probable errors of his measures of distance are somewhat greater than those of his measures of position angle and that both increase with the angular separation of the components, with their faintness, and with the difference in their magnitudes. Modern observers note the same facts in the probable errors of their measures. In their precision, moreover, and in freedom from systematic errors, Struve's measures compare very favorably with the best modern ones.

His observations of star colors show that when the two components of a pair are of about the same magnitude they are generally of the same color, and that the probability of color contrast increases with increasing difference in the brightness of the components, the fainter star being the bluer. Very few exceptions to these results have been noted by later observers.

Finally, in connection with his discussion of the division of double stars into classes by distance, Struve argues, on the theory of probabilities, that practically all the pairs in his first three classes (distance under 4.00″) and the great majority in his first five classes (distance less than 12″) are true binary systems. With increasing angular separation he finds that the probability that optical systems will be included increases, especially among the pairs in which both components are as faint as, or fainter than 8.5 magnitude. This again is in harmony with more recent investigations.

The Russian government now called upon Struve to build and direct the new Imperial Observatory at Pulkowa. Here the principal instruments were an excellent Repsold meridian circle and an equatorial telescope with an object glass of fifteen inches aperture. This was then the largest refractor in the world, as the nine-inch Dorpat telescope had been in 1824.

One of the first pieces of work undertaken with it was a resurvey of the northern half of the sky to include all stars as bright as the seventh magnitude. In all, about 17,000 stars were examined, and the work was completed in 109 nights of actual observing between the dates August 26, 1841, and December 7, 1842. The immediate object was the formation of a list of all the brighter stars, with approximate positions, to serve as a working program for precise observations with the meridian circle. It was thought, however, that the more powerful telescope might reveal double stars which had escaped detection with the nine-inch either because of their small angular separation or because of the faintness of one component. This expectation was fully realized. The survey, which after the first month, was conducted by Wilhelm Struve's son, Otto, resulted in the discovery of 514 new pairs, a large percentage of which were close pairs. These, with Otto Struve's later discoveries which raised the total to 547, are known as the OΣ or Pulkowa double stars. The list of 514 was published in 1843 without measures, and when, in 1850, a corrected catalogue, with measures, was issued, 106 of the original 514 were omitted because not really double, or wider than the adopted distance limits, or for other reasons. But, as Hussey says, "it is difficult effectively to remove a star which has once appeared in the lists." Nearly all of the OΣ stars rejected because of wide separation have been measured by later observers and are retained in Hussey's Catalogue of the OΣ Stars and in Burnham's General Catalogue.

The early period of double star discovery ended with the appearance of the Pulkowa Catalogue. New double stars were indeed found by various observers as incidents in their regular observing which was mainly devoted to the double stars in the great catalogues which have been described and, in particular, to those in the Σ and the OΣ lists. The general feeling, however, was that the Herschels and the Struves had practically completed the work of discovery.

Many astronomers, in the half century from 1820 to 1870, devoted great energy to the accurate measurement of double stars; and the problem of deriving the elements of the orbit of a system from the data of observation also received much attention. This problem was solved as early as 1827, and new methods of solution have been proposed at intervals from that date to the present time. Some of these will be considered in Chapter IV.

One of the most notable of the earlier of these observers was the Rev. W. R. Dawes, who took up this work as early as 1830, using a three and eight-tenths inch refractor. Later, from 1839 to 1844, he had the use of a seven-inch refractor at Mr. Bishop's observatory, and still later, at his own observatory, he installed first a six-inch Merz, then a seven and one-half inch Alvan Clark, and finally an eight and one-half inch Clark refractor. Mr. Dawes possessed remarkable keenness of vision, a quality which earned for him the sobriquet, 'the eagle-eyed', and, as Sir George Airy says, was also "distinguished . . . by a habitual, and (I may say) contemplative precision in the use of his instruments." His observations, which are to be found in the volumes of the *Monthly Notices* and the *Memoirs of the Royal Astronomical Society*, "have commanded a degree of respect which has not often been obtained by the productions of larger instruments."

Another English observer whose work had great influence upon the progress of double star astronomy was Admiral W. H. Smythe, who also began his observing in 1830. His observations were not in the same class with those of Dawes, but his *Bedford Catalogue* and his *Cycle of Celestial Objects* became justly popular for their descriptions of the double and multiple stars, nebulae, and clusters of which they treat, and are still "anything but dull reading."

Far more important and comprehensive than that of any other astronomer of the earlier period after W. Struve was the double star work of Baron Ercole Dembowski who made his first measures at his private observatory near Naples in 1851.

His telescope had an excellent object-glass, but its aperture was only five inches and the mounting had neither driving clock nor position circles. Nor was it equipped with a micrometer for the measurement of position angles; these were derived from measures of distances made in two coordinates. With this instrument Dembowski made some 2,000 sets of measures of high quality in the course of eight years, though how he managed to accomplish it is well-nigh a mystery to observers accustomed to the refinements of modern micrometers and telescope mountings.

In 1859, he secured a seven-inch Merz refractor with circles, micrometer, and a good driving clock, and, in 1862, he resumed his double star observing with fresh enthusiasm. His general plan was to remeasure all of the double stars in the Dorpat and Pulkowa catalogues, repeating the measures in successive years for those stars in which changes were brought to light. His skill and industry enabled him, by the close of the year 1878, to accumulate nearly 21,000 sets of measures, including measures of all of the Σ stars except sixty-four which for one reason or another were too difficult for his telescope. About 3,000 of the measures pertain to the OΣ stars and about 1,700 to stars discovered by Burnham and other observers. Each star was measured on several different nights and for the more interesting stars long series of measures extending over twelve or fifteen or even more years were secured. The comprehensive character of his program, the systematic way in which he carried it into execution, and the remarkable accuracy of his measures combine to make Dembowski's work one of the greatest contributions to double star astronomy. He died before his measures could be published in collected form, but they were later (1883–1884) edited and published by Otto Struve and Schiaparelli in two splendid quarto volumes which are as indispensable to the student of double stars as the *Mensurae Micrometricae* itself.

Mädler at Dorpat, Secchi at Rome, Bessel at Königsberg, Knott at Cuckfield, Engelmann at Leipzig, Wilson and Gledhill at Bermerside, and many other able astronomers published important series of double star measures in the period under consideration. It is impossible to name them all here. Lewis, in his volume on the Struve Stars, and Burnham, in his *General Catalogue* of *Double Stars*, give full lists of the observers, the latter with complete references to the published measures.

CHAPTER II

HISTORICAL SKETCH: THE MODERN PERIOD

The feeling that the Herschels, South, and the Struves had practically exhausted the field of double star discovery, at least for astronomers in the northern hemisphere, continued for thirty years after the appearance of the Pulkowa Catalogue in 1843. Nor were any new lines of investigation in double star astronomy developed during this period. Then, in 1873, a modest paper appeared in the *Monthly Notices of the Royal Astronomical Society*, entitled "Catalogue of Eighty-one Double Stars, Discovered with a six-inch Alvan Clark Refractor. By S. W. Burnham, Chicago, U. S. A."

The date of the appearance of this paper may be taken as the beginning of the modern period of double star astronomy, for to Burnham belongs the great credit of being the first to demonstrate and utilize the full power of modern refracting telescopes in visual observations; and the forty years of his active career as an observer cover essentially all of the modern developments in binary star astronomy, including the discovery and observation of spectroscopic binaries, the demonstration that the 'eclipsing' variable stars are binary systems, and the application of photographic methods to the measurement of visual double stars.

Within a year after the appearance of his first catalogue Burnham had published two additional ones, raising the number of his discoveries to 182. At this time he was not a professional astronomer but an expert stenographer employed as official reporter in the United States Courts at Chicago. He had secured, in 1861, a three-inch telescope with alt-azimuth mounting, and, some years later, a three and three-quarter-inch refractor with equatorial mounting. "This was just good enough," he tells us, "to be of some use, and poor enough . . . to make something better more desirable than ever." In 1870, accordingly, he purchased the six-inch refractor from Alvan Clark and erected it in a small observatory at his home in Chicago. With this instrumental equipment and an astronomical library consisting principally of a copy of the first edition of Webb's *Celestial Objects for Common Telescopes*, Mr. Burnham began his career as a student of double stars. His first new pair (β 40) was found on April 27, 1870.

The six-inch telescope, which his work so soon made famous, was not at first provided with a micrometer and his earliest list of discoveries was printed without measures. Later, position angles were measured but the distances continued to be estimated. This lack of measures by him was covered to a considerable extent by the measures of Dembowski and Asaph Hall.

Burnham's later career has been unique. He has held positions in four observatories, the Dearborn, the Washburn, the Lick, and the Yerkes, and has discovered double stars also with the twenty-six-inch refractor at the United States Naval observatory, the sixteen-inch refractor of the Warner observatory, and the nine and four-tenths-inch refractor at the Dartmouth College observatory. In all, he has discovered about 1,340 new double stars and has made many thousands of measures which are of inestimable value because of their great accuracy and because of the care with which he prepared his observing programs. And yet, except for the two short periods spent respectively at Madison and at Mount Hamilton, he continued his work as Clerk of the United States District Court of Chicago until about eight years ago! He retired from the Yerkes observatory in 1912.

Burnham's plan in searching for new double stars was very different from that followed by his great predecessors. He did not attempt a systematic survey of the sky but examined the stars in a more random way. In his earlier work, while identifying the objects described in Webb's book, he made a practice of examining the other stars near them. Later, whenever he measured a double star, he continued this practice, examining in this manner probably the great majority of the naked eye and brighter telescopic stars visible from our latitudes. Many of the double stars he discovered with the six-inch refractor are difficult objects to measure with an aperture of thirty-six inches. They include objects of two classes almost unrepresented in the earlier catalogues: pairs in which the components are separated by distances as small as 0.2″, and pairs in which one component is extremely faint, and close to a bright primary. In his first two lists he set his limit at 10″, but later generally rejected pairs wider than 5″. The result is that the percentage of very close pairs, and therefore of pairs in comparatively rapid orbital motion, is far higher in his catalogue than in any of the earlier ones.

Burnham's work introduced the modern era of double star discovery, the end of which is not yet in sight. No less distinguished an authority than the late Rev. T. W. Webb, in congratulating Burnham upon his work in 1873, warned him that he could not continue it for any great length of time for want of material. Writing in 1900, Burnham's comment was: "Since that time more than one thousand new double stars have been added to my own catalogues, and the prospect of future discoveries is as promising and encouraging as when the first star was found with the six-inch telescope."

Working with the eighteen and one-half inch refractor of the Dearborn Observatory, G. W. Hough discovered 648 double stars in the quarter-century from 1881 to 1906. In 1896 and 1897, T. J. J. See, assisted by W. A. Cogshall and S. L. Boothroyd, examined the stars in the zone from $-20°$ to $-45°$ declination, and in half of the zone (from 4^h to 16^h R. A.) from $-45°$ to $-65°$ declination with the twenty-four-inch refractor of the Lowell Observatory, and discovered 500 new double stars. See states that not less than 100,000 stars were examined, "many of them, doubtless, on several occasions." This is probably an overestimate for it leads to a remarkably small percentage of discoveries.

In England, in 1901, the Rev. T. E. H. Espin began publishing lists of new double stars discovered with his seventeen and one-fourth inch reflector. The first list contained pairs casually discovered in the course of other work; later, Mr. Espin undertook the systematic observation of all the stars in the Bonn Durchmusterung north of +30°, recording, and, as far as possible measuring, all pairs under 10″ not already known as double. At this writing, his published discoveries have reached the total of 1,356.

In France, M. Robert Jonckheere began double star work in 1909 at the Observatoire D'Hem and has discovered 1,319 new pairs to date. Since 1914, he has been at Greenwich, England, and has continued his work with the twenty-eight-inch refractor. The majority of his double stars, though close, are quite faint, a large percentage of them being fainter than the 9.5 magnitude limit of the Bonn Durchmusterung.

Shorter lists of discoveries have been published by E. S. Holden, F. Küstner, H. A. Howe, O. Stone, Alvan and A. G. Clark, E. E. Barnard, and others, and many doubles were first noted by the various observers participating in the preparation of the great Astronomische Gesellschaft Catalogue.

My own work in this field of astronomy began when I came to the Lick Observatory in June, 1895. At first my time was devoted to the measurement with the twelve-inch refractor of a list of stars selected by Professor Barnard, and the work was done under his direction. Later, longer lists were measured both with this telescope and with the thirty-six-inch refractor; and in selecting the stars for measurement I had the benefit of advice—so generously given by him to many double star observers of my generation—from Professor Burnham, then at the Yerkes Observatory. My attention was early drawn to questions relating to double star statistics, and before long the conviction was reached that a prerequisite to any satisfactory statistical study of double star distribution was a re-survey of the sky with a large modern telescope that should be carried to a definite limiting magnitude. I decided to undertake such a survey, and, adopting the magnitude 9.0 of the Bonn Durchmusterung as a limit, began the preparation of charts of convenient size and scale showing every star in the B. D. as bright as 9.0 magnitude, with notes to mark those already known to be double. The actual work of comparing these charts with the sky was begun early in April, 1899.

Professor W. J. Hussey, who came to the Lick Observatory in January, 1896, also soon took up the observation of double stars. His first list consisted of miscellaneous stars, but, in 1898, he began the remeasurement of all of the double stars discovered by Otto Struve, including the 'rejected' pairs. This work was carried out with such energy and skill that in 1901, in Volume V of the Lick Observatory Publications, a catalogue of the OΣ stars was published which contained not only Hussey's measures of every pair but also a complete collection of all other published measures of these stars, with references to the original publications, and discussions of the motion shown by the various systems. In the course of this work, Hussey had found an occasional new double star and had decided that at its conclusion he would make more thorough search for new pairs. In July, 1899, we accordingly combined forces for the survey of the entire sky from the North Pole to $-22°$ declination on the plan which I had already begun to put into execution; Hussey, however, charted also the 9.1 B.D. stars. Each observer undertook to examine about half the sky area, in zones $4°$ wide in declination. When Mr. Hussey left the Lick Observatory in 1905, I took over his zones in addition to those assigned to me in our division of the work and early in 1915 completed the entire survey to $-22°$ declination, as originally planned, between 13^h and 1^h in right ascension, and to $-14°$ declination in the remaining twelve hours. These come to the meridian in our winter months when conditions are rarely satisfactory for work at low altitudes. To complete the work to $-22°$ in these hours would require several years.

The survey has resulted in the discovery of more than 4,300 new pairs, 1,329 by Hussey, the others by me, practically all of which fall within the distance limit of $5''$. The statistical conclusions which I have drawn from this material will be presented in a later chapter.

It may seem that undue emphasis has been placed upon the *discovery* of double stars in this historical sketch. That a particular star is or is not double is indeed of relatively little consequence; the important thing is to secure accurate measures through a period of time sufficiently long to provide the data for a definite determination of the orbit of the system. But the discovery must precede the measures, as Sir John Herschel said long ago; moreover, such surveys as that of Struve and the one recently completed at the Lick Observatory afford the only basis for statistical investigations relating to the number and spatial distribution of the double stars. Further, the comparison of the distance limits adopted by the successive discoverers of double stars and an analysis of the actual distances of the pairs in their catalogues affords the most convenient measure of the progress made in the 140 years since Herschel began his work, both in the power of the telescopes available and in the knowledge of the requirements for advance in this field of astronomy.

The data in the first four lines of the following table are taken from Burnham's General Catalogue of his own discoveries, and in the last two lines I have added the corresponding figures for the Lick Observatory double star survey, to 1916.

The Percentage of Close Pairs in Certain Catalogues of Double Stars:

	Class I Number of Stars	Class II Number of Stars	Sum	Percentage of Close Pairs
William Herschel, Catalogue of 812 Stars	12	24	36	4.5
Wilhelm Struve, Catalogue of 2,640 Stars	91	314	405	15.0
Otto Struve, Catalogue of 547 Stars	154	63	217	40.0
Burnham, Catalogue of 1,260 Stars	385	305	690	55.0
Hussey, Catalogue of 1,327 Stars	674	310	984	74.2
Aitken, Catalogue of 2,900 Stars	1,502	657	2,159	74.4

The increasing percentage of close pairs is of course due in part to the earlier discovery of the wider pairs, but the absolute numbers of the closer pairs testify to the increase of telescopic power in the period since 1780. If Class I had been divided into two sub-classes including pairs under 0.50″ and pairs between 0.51″ and 1.00″, respectively, the figures would have been even more eloquent, for sixty per cent. of the Class I pairs in the last two Catalogues enumerated have measured distances of 0.50″ or less.

While the modern period is thus characterized by the number of visual binaries, and, in particular, those of very small angular distance discovered within it, it is still more notable for the development of an entirely new field in binary star astronomy. In August, 1889, Professor E. C. Pickering announced that certain lines in the objective-prism spectrograms of ζ *Ursae Majoris* (*Mizar*) were double on some plates, single on others, the cycle being completed in about 104 days.[1] An

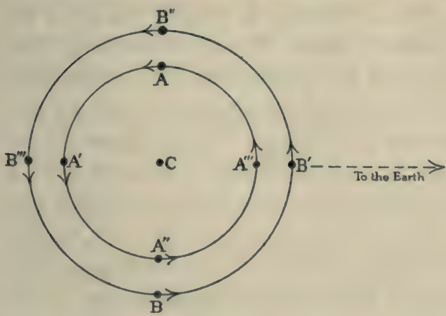

FIGURE 1. A, A′, A″, A‴ = primary star at points of maximum, minimum and mean radial velocity.
B, B′, B″, B‴ = position of the companion star at the corresponding instants.
C is the center of gravity of the system. There is no star or other body at this point.

explanation of the phenomenon was found in the hypothesis that the star consisted of two components, approximately equal in brightness, in rapid revolution about their center of mass.

If the orbit plane of such a system is inclined at a considerable angle to the plane of projection, the velocities in the line of sight of the two components will vary periodically, as is evident from Figure 1; and, on the Doppler-Fizeau principle,[2] there will be a slight displacement of the lines of the spectrum of each component from their mean positions toward the violet end when that component is approaching the Earth, relatively to the motion of the center of mass of the system, and toward the red end when it is receding, relatively. It is clear from the figure that when one component is approaching the Earth, relatively, the other will be receding, and that the lines of the two spectra at such times will be displaced in opposite direc-

[1] The real period, deduced from many plates taken with slit-spectrographs, is about one-fifth of this value, a little more than 20.5 days.

[2] Explained in Chapter V.

tions, thus appearing double on the spectrograms. Twice, also, in each revolution the orbital motion of the two components will evidently be directly across the line of sight and the radial velocity of each at these times is the same, and is equal to that of the system as a whole. The lines of the two spectra, if similar, will then coincide and appear single on the plates. There is no question but that this explanation is the correct one, and *Mizar* therefore has the honor of being the first star discovered to be a spectroscopic binary system.

A moment's consideration is enough to show that if one of the two components in such a system is relatively faint or 'dark' only one set of spectral lines, that produced by the brighter star, will appear upon the plate, but that these will be shifted periodically from their mean positions just as are the lines in the double spectrum of *Mizar*. If the plane of the system lies so nearly in the line of sight that each star partly or completely eclipses the other once in every revolution the presence of the darker star may be revealed by a periodic dimming of the light of the brighter one; if the orbit plane, as will more commonly happen, is inclined at such an angle to the line of sight that there is no occultation or eclipse of the stars for observers on the Earth the variable radial velocity of the brighter star will be the sole evidence of the existence of its companion.

Algol (*β Persei*) is a variable star whose light remains nearly constant about four-fifths of the time; but once in every two and one-half days it rapidly loses brightness and then in a few hours' time returns to its normal brilliancy. As early as 1782, Goodericke, the discoverer of the phenomenon, advanced the theory that the periodic loss of light was due to the partial eclipse of the bright star by a (relatively) dark companion. In November, 1889, Professor Hermann Vogel, who had been photographing the spectrum of the star at Potsdam, announced that this theory was correct, for his spectrograms showed that before light minimum the spectral lines were shifted toward the red from their mean position by an amount corresponding to a velocity of recession from the Earth of about twenty-seven miles a second. While the star was recovering its brightness, on the other hand, the shift of the lines toward the violet indicated a somewhat greater velocity of approach, and the period of revolution determined by means of the curve plotted from the observed radial velocities was identical with the period of light variation. *Algol* thus became the second known spectroscopic binary star and the first of the special class later called eclipsing binaries.

Within a few months two other spectroscopic binary stars were discovered; *β Aurigae* by Miss Maury at the Harvard College Observatory from the doubling of the lines in its spectrum at intervals of slightly less than two days (the complete revolution period is 3.96 days), and *α Virginis*, by Vogel. The latter star was not variable in its light, like *Algol*, nor did its spectrum show a periodic doubling of the lines,[3] like *Mizar* and *β Aurigae*, but the lines of the single spectrum were displaced periodically, proving that the star's radial velocity varied, and the cycle of variation was repeated every four days. *α Virginis* is thus the first representative of that class of spectroscopic binary systems in which one component is relatively dark, as in the case of *Algol*, but in which the orbit plane does not coincide even approximately with the line of sight. It is to this class that the great majority of spectroscopic binary stars now known belong. The reader must not infer that the companion stars in systems of this class emit no light; the expression *relatively* dark may simply mean that the companion is two or three magnitudes fainter than its primary. If the latter were not present, the companion in many systems would be recognized as a bright star; even the companion of *Algol* radiates enough light to permit the secondary eclipse, when the primary star is the occulting body, to be detected by our delicate modern photometers.

The story of the modern spectrograph and its revelations of the chemical composition of the stars and nebulae and of the physical conditions which prevail in them is a marvelous one, but this is not the place to tell it. We must limit ourselves to the simple statement that in the years since 1889 the spectrograph has also given us a vast amount of information with

[3]The secondary spectrum of *α Virginis* has been photographed in more recent years.

regard to the radial velocities of the stars and, as a by-product, with regard to spectroscopic binary systems. In this development the Lick Observatory has taken a leading part, for by the application of sound engineering principles in the design of the Mills spectrograph, and by patient and skilful experimental work extended over several years, Dr. Campbell was enabled, in the late 1890's, to secure an accuracy of measurement of radial velocity far surpassing any previously attained. The New Mills spectrograph, mounted in 1903, led to even better results, and it is now possible, in the more favorable cases, to detect a variation in the radial velocity even if the range is only one and one-half kilometers per second. Other observers and institutions have also been most active and successful, and the number of known spectroscopic binaries has increased with great rapidity. The *First Catalogue of Spectroscopic Binaries*, compiled by Campbell and Curtis to include the systems observed to January 1, 1905, had 140 entries; by January 1, 1910, when Dr. Campbell prepared his *Second Catalogue of Spectroscopic Binary Stars*, the number had grown to 306, and the Third Catalogue, now in preparation, will contain at least 596 entries.

A preliminary count of the last named Catalogue results in the following table which gives the distribution of these discoveries by observatories:

Lick Observatory, Mount Hamilton	186
D. O. Mills Station of the Lick Observatory, at Santiago, Chile	146
Yerkes Observatory, at Williams Bay	134
Solar Observatory, Mount Wilson	70
Other Observatories in the United States and in Canada	33
European Observatories	27
	596

A slightly different distribution will doubtless result from the final count but the table clearly shows that American observatories have made this field of research peculiarly their own. The Pulkowa, Potsdam, and Bonn observatories are the three in Europe which are giving most attention to the measurement of stellar radial velocities.

The discoveries of the spectroscopic binary stars are here credited to observatories rather than individuals because it is often a matter for fine discrimination to decide with whom the credit for a particular discovery should rest. In general, at least three spectrograms are required to prove that a star is a spectroscopic binary star. These may all be taken and measured by a single observer, or the three plates may be exposed by as many different observers in the course of carrying out a program of work planned by a fourth; the plates may be measured by one or more of the four or by others; variation in the radial velocity may be suspected from the second plate and confirmed by the third or only by a fourth or still later plate. The program for stellar radial velocity determination for the Lick Observatory and its auxiliary station, the D. O. Mills Observatory, in Chile, for example, is planned, and its execution supervised by Dr. Campbell; Wright, Curtis, Moore, R. E. Wilson, Burns, Paddock, and perhaps a score of Fellows and Assistants have been associated with him in the actual work. In his three catalogues of Spectroscopic Binaries, Campbell credits the discovery of the spectroscopic binary stars found in the course of this work to the individual who detected the variation in radial velocity from his measures of the plates.

The problem of finding the elements of the orbit of a spectroscopic binary from the data given by the measures of radial velocity was solved as early as 1891 by Rambaut, and in 1894, Lehmann-Filhés published the method which has been the chief one used ever since. A number of other methods have been proposed in more recent years, some analytical, others graphical, and doubtless others still will be developed. This phase of the subject is treated in Chapter VI.

At the present time orbits for 137 systems have been computed, a number exceeding that of the visual binary systems with known orbits. The reason is not far to seek. The visual binaries are systems of vast dimensions and their revolution periods range from a minimum (so far as known at present) of five and seven-tenths years to a maximum that is certainly greater than 500 years and that may exceed a thousand. *Castor*, for example, was one of the first double stars to be observed, and it was the one in which the fact of orbital motion was first demonstrated; but although the observations extend from the year 1719 to date, the length of the revolution period is still quite uncertain. The spectroscopic binary stars, on the other hand, are, in general, systems of relatively small dimensions, the revolution periods ranging from five or six hours, as a minimum, to a few years. The masses of the systems being assumed to be of the same order, the smaller the dimensions, the greater the orbital velocity, and the greater the probability of the detection of the system by means of the spectrograph, for the amount of the displacement of the lines in the spectrum is a function of the radial velocity of the star.

Now if the revolution of a system is accomplished in, say two or three days, it is obviously possible for an observer to secure ample data for the computation of its definitive orbit in a single season. Indeed, if the spectrograph is devoted to this purpose exclusively and the percentage of clear nights is large, a single telescope may in one season secure the data for the orbits of twenty or more systems.

As in the work of their discovery, so in the computation of the orbits of the spectroscopic binary stars the American observatories are taking the lead. The Dominion Observatory, at Ottawa, Canada, is making a specialty of this phase of the work and its observers, notwithstanding the handicap of a none too favorable climate for observing work, have more orbits to their credit than those of any other institution. The Lick and the Allegheny observatories follow quite closely, and the Yerkes and Detroit observatories have made valuable contributions.

While the spectroscopic binary stars have been receiving ever increasing attention in recent years, the visual binary stars are by no means being neglected. The work of measuring and remeasuring the double stars discovered has been carried on enthusiastically by scores of able observers with small telescopes and with large ones. It is impossible to comment upon all of these or to give details of the hundreds of series of measures they have published. But I cannot refrain from referring here to two of the most prominent observers of the generation that has just passed away—G. V. Schiaparelli and Asaph Hall. Schiaparelli's measures are published in two quarto volumes, the first containing the measures made at Milan with the eight-inch refractor, in the years 1875 to 1885; the second, the series made with the eighteen-inch refractor at the same observatory in the interval from 1886 to 1900. Hall's work, carried out with the twenty-six-inch refractor of the United States Naval Observatory at Washington, is also printed in two quarto volumes, the first containing the measures made in the years 1875 to 1880; the second, those made from 1880 to 1891. The working lists of both observers were drawn principally from the Dorpat and Pulkowa catalogues, but include many of Burnham's discoveries and some made by Hough and by others. The high accuracy of their measures and the fact that they—and Schiaparelli in particular—repeated the measures of the more interesting stars year after year makes the work of these observers of the greatest importance.

At present, double stars are regularly measured at a number of the largest observatories of this country, at several important observatories in England and on the continent of Europe, and by many enthusiastic amateurs in this country and abroad. So voluminous is the literature of the subject that one who wishes to trace the full record of one of the double stars discovered by Herschel or by Struve in the original sources must have access to a large astronomical library. This condition was recognized many years ago, and as early as 1874 Sir John Herschel's "A Catalogue of 10,300 Multiple and Double Stars, Arranged in Order of R. A." was published as a Memoir of the Royal Astronomical Society. This catalogue attempted merely to give a consecutive list of the known double stars, without measures, and did not go far towards meeting the needs of observers or computers. The first really serviceable compendium was that published by Flammarion in 1878, entitled "Catalogue des Étoiles Doubles et Multiples en Mouvement relatif certain." The volume aimed to include all pairs known from the actual measures to be in motion; 819

systems are listed, each with a fairly complete collection of the published measures, about 14,000 in all, and notes on the nature of the motion. For thirty years this book formed a most excellent guide to observers.

The following year, 1879, "A Handbook of Double Stars," prepared by Crossley, Gledhill, and Wilson, was published in London—a work that has had a wide circulation and that has proved of the greatest service to students of double star astronomy. It is divided into three parts, the first two giving a general account of double star discoveries and methods of observing and of orbit computation. The third section contains a "Catalogue of 1,200 double stars and extensive lists of measures." An appendix gives a list of the principal papers on double stars.

In 1900, Burnham published a General Catalogue of his own discoveries containing a complete collection of all known measures of these stars with notes discussing the motion when such was apparent, and references to the original sources from which the measures were taken. This proved to be the first of a series of such volumes. Hussey's catalogue of the Otto Struve stars, to which reference has already been made, was published in 1901, and five years later, in 1906, Lewis's great volume on the Struve stars appeared. This is, in effect, a revision of the *Mensurae Micrometricae* and gives all of the Σ stars in the order of their original numbers, disregarding the inconvenient division into classes. Such of the Σ 'rejected' stars as have been measured by later observers are also included, and all or nearly all of the published measures of each pair. The notes give an analysis and discussion of the motions which have been observed, and form one of the most valuable features of the work, for the author has devoted many years to a comprehensive study of double star astronomy in all its phases. In 1907, Eric Doolittle published a catalogue of the Hough stars, all of which he had himself reobserved, and in 1915, Fox included in Volume I of *The Annals of the Dearborn Observatory* catalogues of the discoveries of Holden and of Küstner with a new series of measures of these stars. Thus all of the longer catalogues of new double stars, except the very recent ones and those of Sir John Herschel, have now been revised and brought up to date, for Sir William Herschel's discoveries, except the very wide pairs, are practically all included in the *Mensurae Micrometricae*.

Every one of the volumes named is most convenient for reference and each one contains information not easily to be found elsewhere; but they are all surpassed by Burnham's comprehensive and indispensable work, *A General Catalogue of Double Stars within 121° of the North Pole*, which was published by The Carnegie Institution of Washington in 1906. This monumental work consists of two parts, printed in separate quarto volumes. Part I contains a catalogue of 13,665 double stars, including essentially every pair, close or wide, within the sky area named, that had been listed as a double star before 1906. The positions, for 1880, are given, with the discovery date and measure or estimate. Part II contains measures, notes and complete references to all published papers relating to each pair. This great work in itself is an ample guide to anyone who wishes to undertake the measurement of double stars and desires to give his attention to those pairs, not very recent discoveries, which are most in need of observation. When Burnham retired from active astronomical work, he turned over to Professor Eric Doolittle, of the Flower Observatory, all material he had accumulated since the appearance of the General Catalogue; and Doolittle has since kept a complete record of every published measure and orbit, with the view of printing an extension to the catalogue when the need for it is manifest.[4]

It has been convenient, in this narrative, to confine attention until now to the double star work done at observatories in the northern hemisphere, for it has been there that this

[4] M. Robert Jonckheere has just published, in the *Memoirs of the Royal Astronomical Society* (vol. lxi, 1917), a *Catalogue and Measures of Double Stars discovered visually from 1905 to 1916 within 105° of the North Pole and under 5″ Separation*. This is, in effect, an extension of Burnham's *General Catalogue*, though the author has excluded pairs wider than 5″ instead of recording every pair announced by its discoverer as double and has adopted a more northern sky limit than Burnham's. The volume is particularly valuable because it gives in collected form Jonckheere's own discoveries with measures at a second epoch as well as at the time of discovery. The other long lists in the volume are Espin's discoveries and those made at the Lick Observatory; in all, there are 3,950 entries, sufficient evidence of the activity of double star discoverers in recent years.

branch of astronomy has received most attention. Even today there are relatively few telescopes in the southern hemisphere and only two or three of these are in use in the observation of double stars. But the state of our knowledge of the southern double stars is better than this fact would indicate. Many stars south of the equator have been discovered from stations in the northern hemisphere, and the few southern workers in this field have made a most honorable record.

We have seen that two of the earliest double stars discovered—α *Centauri* and α *Crucis*—were stars not visible from European latitudes; but the first extensive list of double stars collected at a southern observatory was James Dunlop's catalogue of 253 pairs observed at Parametta, N. S. W., in the years 1825–1827 with a nine-foot reflecting telescope. These stars, however, are as a rule very wide pairs and are of comparatively little interest. A few double stars are contained in Brisbane's Parametta catalogue, published in 1835, and more in the later meridian catalogues of the Royal Observatory at the Cape of Good Hope, the Argentine National Observatory at Cordoba, and of other southern observatories.

The most important early paper on southern double stars is beyond question the chapter upon them in Sir John Herschel's *Results of Astronomical Observations made during the Years 1834, 1835, 1836, 1837, 1838 at the Cape of Good Hope* which was published in 1847. Innes says, "The sections on double stars in this work are to the southern heavens what Struve's *Mensurae Micrometricae* are to the northern heavens." A catalogue of the discoveries made at Feldhausen, C. G. H., with the twenty-foot reflector is given, which contains the pairs h3347 to h5449, together with measures of such previously known pairs as were encountered in the 'sweeps'. Many of the new pairs are wide and faint, resembling the h stars discovered at Slough, in England; but many others are comparatively close, many are very bright, and a number are among the finest double stars in the southern sky. Another division of this chapter gives the micrometer measures, made with the five-inch refractor, of many of these new pairs and of some of the known ones. Innes says that "the angles of the pairs are all through of high excellence"; but Herschel himself points out the sources of weakness in his measures of distances.

Herschel's station at Feldhausen was not a permanent observatory, and when he returned to England work there was discontinued; nor was double star work seriously pursued at any other southern station until about forty years later. In 1882, a list of 350 new pairs was published by H. C. Russell, director of the Sydney Observatory, N. S. W., the measures being made by Russell and by L. Hargrave. In 1884, an additional list of 130 pairs, mostly wide, was published, and in the following years several lists of measures by these observers and their colleague, J. A. Pollock, a few of the measured pairs being new. In 1893, R. P. Sellors published a short list (fourteen pairs, all under 2″) discovered by him at the same observatory, and in the following years he contributed many measures of known pairs and discoveries of a few additional new ones.

The man of the present generation who has done most to advance double star astronomy in the southern hemisphere is R. T. A. Innes, now Government Astronomer at the Union Observatory, Johannesburg, Union of South Africa. In 1895, he published a list of twenty-six pairs 'probably new' which were found with a six and one-fourth-inch refractor at Sydney, N. S. W., and the following year, sixteen additional discoveries made with a small reflector. In this year, 1896, Mr. Innes joined the staff of the Royal Observatory of the Cape of Good Hope and there, in addition to his regular duties, continued his double star work with the seven-inch refractor, and later, for a time, with the eighteen-inch McClean refractor. With these instruments he brought the total of his discoveries to 432 and made a fine series of measures. Since going to his present station in 1903, he has discovered more than 600 additional pairs with a nine-inch refractor and has made extensive series of measures which are of the greatest importance not alone because of the stars measured but also because the work has been most carefully planned to eliminate systematic errors of measure as far as possible. A large modern refractor is to be erected at the Union Observatory as soon as the glass disks can be secured, and it is Mr. Innes's intention to use this instrument in an even more systematic study of double stars.

Another telescope was set to work upon southern double stars when Professor Hussey accepted the directorship of the observatory of the La Plata University, Argentina, in 1911, in addition to his duties at Ann Arbor, Michigan. Mr. Hussey has spent several periods at La Plata organizing the work of the observatory and personally using the seventeen-inch refractor in searching for, and measuring double stars. So far the work has resulted in the publication of two lists of new pairs containing 312 stars and of a valuable list of measures of known pairs.

Mr. Innes has shown that even before he began his own work the number of close double stars known in the sky area south of $-19°$ declination exceeded the number in corresponding distance classes north of $+19°$ declination contained in the *Mensurae Micrometricae*. His own discoveries and those by Hussey at La Plata consist almost entirely of close pairs, and we may allow his claim, without serious protest, that in point of double star *discovery* the southern hemisphere is not greatly in arrears. If Innes and Hussey are able to carry out their programs for systematic surveys of the sky to the South Pole, we shall have as complete data for statistical studies of the southern double stars as we now have for those of the northern pairs.

In 1899, Innes published his *Reference Catalogue of Southern Double Stars* which has proved to be a most valuable work. The object was to include "all known double stars having southern declination at the equinox of 1900"; but the author did not follow the plan adopted by Burnham of including all objects published as double stars regardless of angular separation. Instead, he adopted limits which varied with the magnitudes of the stars, ranging from 1″ for stars of the ninth magnitude to 30″ for those of the first magnitude. This course has been sharply criticized by some writers, but I think there can be no serious question as to the soundness of the principle involved. Whether the limits actually adopted are those best calculated to promote the progress of double star astronomy is a different matter and raises a question to which more attention will be given on a later page. Mr. Innes has in preparation a new edition of the Reference Catalogue, bringing it up to the present date. Possibly he may adopt in it some modifications of his former limits. A feature of the work to which reference must be made is the excellent bibliography of double star literature which forms the appendix.

Our knowledge of the spectroscopic binary stars in the far southern skies is due almost entirely to the work carried on at the D. O. Mills Station of the Lick Observatory, established at Santiago, Chile, in 1893. The instrumental equipment consists of a thirty-seven and one-fourth-inch silver-on-glass reflector and spectrographs similar in design to those in use on Mount Hamilton. The working program is the measurement of the radial velocities of the stars and nebulae which are too far south to be photographed at the Lick Observatory itself. The discovery of binary stars is not the object in view, but the table given on page 30 shows that more than one-fifth of the entire number of these systems known at the present time have been found at this Station in the fourteen years of its existence. When we add to this number the spectroscopic binary stars with southern declinations which have been detected by observers at stations in the northern hemisphere, we shall find that in this field there is no disparity whatever between the two hemispheres of the sky.

CHAPTER III

OBSERVING METHODS, VISUAL BINARY STARS

The operation of measuring a double star is a very simple one. The object is to define at a given instant the position of one star, called the *companion*, with respect to the other, known as the *primary*. When the two stars are of unequal magnitude the brighter is chosen as the primary; when they are of equal brightness, it is customary to accept the discoverer's designations.

From the first work by Sir William Herschel, the measures have been made in polar coordinates; and since about 1828, when Sir John Herschel recommended the practice, the *position angle* has been referred to the North Pole as zero point and has been counted through 360°.

That is, the *position angle* is the angle at the primary star between the line drawn from it to the North Pole and one drawn from it to the companion, the angle increasing from zero when the companion is directly north through 90° when it is at the east, 180° when it is south, 270° when it is west, up to 360° when it is once more directly north. The *distance* is the angular separation between the two stars measured at right angles to the line joining their centers. The two coordinates are usually designated by the Greek letters θ and ρ, or by the English letters p and s.

THE MICROMETER

Experience has proved that the parallel-wire micrometer is the best instrument for double star measurements. A complete description of it is not necessary here. For this the reader is referred to Gill's article on the Micrometer in the *Encyclopedia Britannica*. Essentially it consists of a tube or adapter firmly fitted into the eye-end of the telescope and carrying on its outer end a graduated circle (the position circle) reading

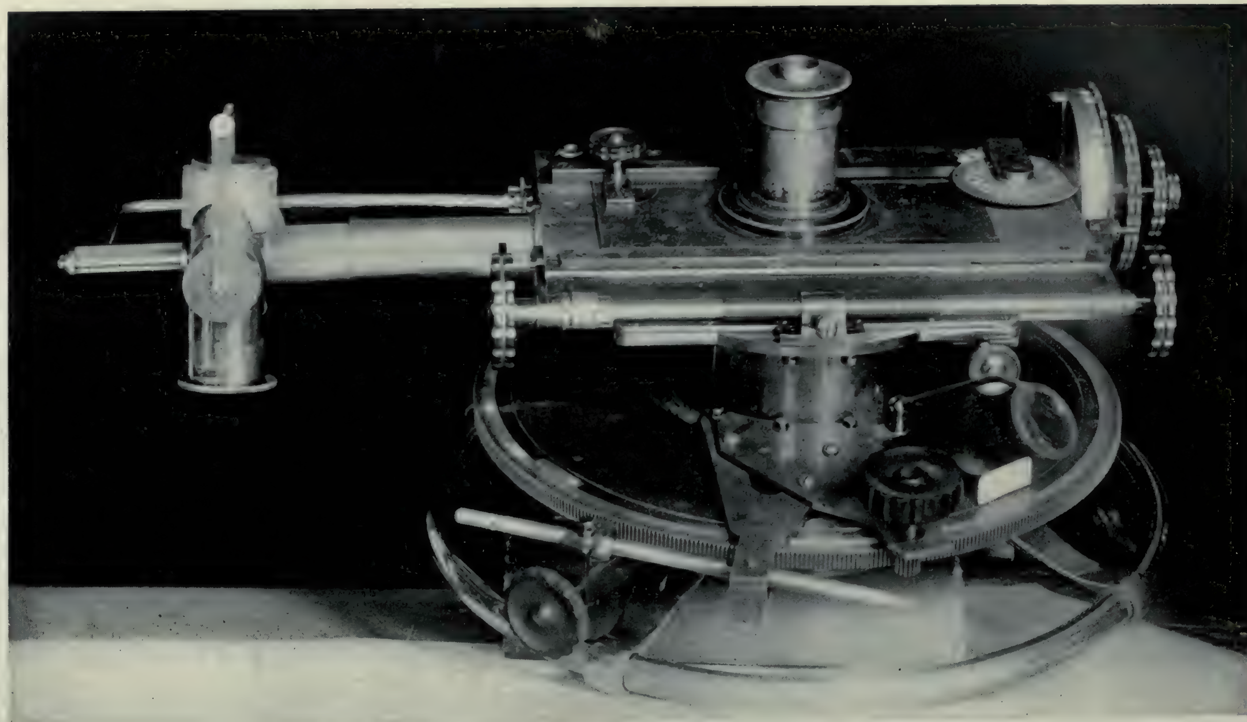

Plate II. The Micrometer for the Thirty-Six-Inch Refractor from 0° to 360° in a direction contrary to the figures on a clock dial. A circular plate fitting closely within the position circle and adjusted to turn freely within it carries an index, or a vernier, or both, to give the circle reading. In the micrometers in use at the Lick Observatory, this plate is rotated about the optical axis of the telescope by an arm carrying a pinion which meshes into rack teeth cut on the outer circumference of the position circle. A clamp is provided to hold the plate and circle together at any desired reading, and a tangent screw to give a slow motion. Upon the vernier plate an oblong box is mounted within which the parallel wires or threads (they are usually spider lines) are placed. This box is movable longitudinally by a well-cut, but not very fine screw. One thread, the *fixed* thread, is attached to the inner side of the upper plate of the box, and the other, the *micrometer* or *movable* thread, is attached to a frame or fork which slides freely in the box longitudinally, but without any lateral play. The fork is moved by a very fine and accurately cut screw which enters the box at one end. At its outer extremity, this screw carries a milled head divided into 100 parts, the readings increasing as the screw draws the micrometer thread towards the head. Strong springs at the opposite end of the fork carrying this thread prevent slack, or lost motion.

The two threads, the fixed and the micrometer, must be so nearly in the same plane—the focal plane of the objective—that they can be brought into sharp focus simultaneously in an eye-piece of any power that may be used, but at the same time must pass each other freely, without the slightest interference. Instead of a single fixed thread, some micrometers carry systems of two, three, or more fixed threads, and frequently also one or more fixed transverse threads. Some also substitute two parallel threads separated a few seconds of arc for the single movable thread. For double star work, the simple micrometer with only two threads is unquestionably to be preferred, and even for comet, asteroid, satellite, and other forms of micrometric work, I regard it as superior to the more complicated forms and less liable to lead to mistakes of record.

The telescope is assumed to be mounted stably and to be in good adjustment. Assured as to these two points and as to the firm attachment of the micrometer to the telescope tube that the zero reading of his position circle shall remain constant, the double star observer has still to determine the value of one revolution of his micrometer screw and the zero or north point reading of his position circle before beginning actual measurement. The reading for coincidence of the threads is eliminated by the method of double distance measures, as will be shown presently, and the distances themselves are, in general, so small, and modern screws so accurate, that irregularities in the screw and corrections for temperature may be regarded as negligible. If desired, however, they may be determined in connection with measures for the revolution value.

THE ZERO POINT

The determination of the zero point will be considered first. The simplest practical method, and the one adopted by observers generally, is to put on the lowest power eye-piece which utilizes the entire beam of light, direct the telescope upon an equatorial star near the meridian, stop the driving-clock, and turn the micrometer by the box screw and the position-circle pinion until the star 'trails' along the thread across the entire field of view. The star should be bright enough to be seen easily behind the thread, but not too bright. With the twelve-inch telescope I find a star of the seventh or eighth magnitude most satisfactory; with the thirty-six-inch, one of the ninth or tenth magnitude. A little practice will enable the observer to determine his 'parallel' reading with an uncertainty not greater than one-fifth of one division of his circle. On the micrometer used with the thirty-six-inch telescope, this amounts to 0.05°. Several independent determinations should be made. If the micrometer is not removed from the telescope and is set firmly to the tube, it is probable that the parallel reading need be checked only once or twice a week. When, as at the Lick Observatory, the micrometer is liable to be removed almost any day and is certainly removed several times every week, the observer very promptly forms the habit of determining the parallel at the beginning of his work *every night;* my own practice is to check the value at the close of work also.

90° added to the parallel gives the north point or zero reading.

REVOLUTION OF THE MICROMETER SCREW

The value of the revolution of the micrometer screw should be determined with the greatest care and the investigation should be repeated after a reasonable time interval to detect any wear of the screw. Two different methods of procedure are about equally favored by observers: the method of transits of circumpolar stars and the method of direct measures of the difference in declination of suitable pairs of stars.

In the first method the position circle is set for the *zero* reading (*i. e.*, 90° from the reading for parallel) and the telescope turned upon the star a short time before it culminates. (The driving clock, of course, is stopped.) Set the micrometer thread just in advance of the star as it enters the field of view (it is convenient to start with the milled head set at zero of a revolution) and note the time of the star's transit either on the chronograph or by the eye-and-ear method. Advance the thread one revolution or a suitable fraction of a revolution and take another transit, and repeat this procedure until the star has crossed the entire field of view. A low-power eye-piece should be used and the series of measures so planned that they will extend over from forty to eighty revolutions of the screw, about half of the transits being taken before the star crosses the meridian, the other half after. Great care must be taken not to disturb the instrument during the course of the observations for the slightest changes in its position will introduce errors into the measures. It is well to repeat the observations on a number of nights, setting the telescope alternately east and west of the pier. A sidereal time piece should be used in recording the times of transits and if it has a large rate, it may be necessary to take this into account. Theoretically, a correction for refraction should also be introduced, but if all of the measures are made near the meridian, this correction will rarely be appreciable.

In the figure, let P be the pole, EP the observer's meridian, ab the diurnal path of a star, AS the position of the micrometer thread when at the center of the field and parallel to an hour circle PM, and BS' any other position of the thread. Now let $m_\circ$ be the micrometer reading, $t_\circ$ the hour angle, and $T_\circ$ the sidereal time when the star is at S, and m, t, and T the corresponding quantities when the star is at S', and let R be the value of one revolution of the screw.

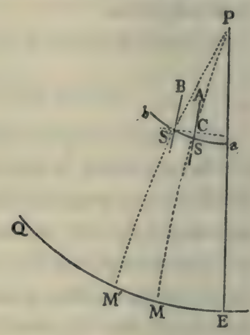

FIGURE 2

Through S' pass an arc of a great circle $S'C$ perpendicular to AS. Then, in the triangle $CS'P$, right-angled at C, we have

$$CS' = (m - m_\circ)R, \quad S'P = 90^\circ - \delta, \quad CPS' = t - t_\circ = T - T_\circ;$$

and we can write

$$\sin [(m - m_\circ)R] = \sin (T - T_\circ) \cos \delta; \qquad (1)$$

or, since $(m - m_\circ)R$ is always small,

$$(m - m_\circ) R = \sin(T - T_\circ) \cos \delta / \sin 1''. \qquad (2)$$

Similarly, for another observation,

$$(m' - m_\circ) R = \sin(T' - T_\circ) \cos \delta / \sin 1''. \qquad (3)$$

Combining these to eliminate the zero point,

$$(m' - m)R = \sin(T' - T_\circ) \cos \delta / \sin 1'' - \sin(T - T_\circ) \cos \delta / \sin 1'' \quad (4)$$

from which the value of R is obtained. The micrometer readings are supposed to increase with the time.[1]

If eighty transits have been taken, it will be most convenient to combine the first and the forty-first, the second and the forty-second, and so on, and thus set up forty equations of condition of the form of equation (4). The solution of these equations by the method of least squares will give the most probable value for R.

If the value of R is to be determined by direct measures of the difference of declination between two stars, the stars should satisfy the following conditions: they should lie on,

[1] From Campbell's *Practical Astronomy*.

or very nearly on, the same hour circle; their proper motions as well as their absolute positions at a given epoch should be accurately known; they should be nearly of the same magnitude and, if possible, of nearly the same color; the difference of declination should amount to from fifty to one hundred revolutions of the micrometer screw; and, since this will ordinarily exceed the diameter of the field of view of the eye-piece, one or more intermediate stars (whose positions do not need to be so accurately known) should lie nearly on the line joining them and at convenient intervals to serve as steps.

There are not many pairs of stars which answer all of the requirements. Probably the most available ones are to be found in the *Pleiades* and other open clusters which have been triangulated by heliometer observations.

The measures should be made only on the most favorable nights and at times when the stars are high enough in the sky to make the correction for refraction small. The difference of declination should be measured from north star to south star and also in the opposite direction and the measures should be repeated on several nights. If extreme accuracy is desired in the refraction corrections the thermometer and barometer should be read at the beginning and also at the end of each set of measures, and if the effect of temperature is to be included in the determination of R, measures must be made at as wide a range of temperature as is practicable.

In making the reductions, the star-places are first brought forward from the catalogue epoch to the date of the actual observations by correcting rigorously for precession, proper motion, and the reduction from mean to apparent place. The apparent place of each star must then be corrected for refraction. It will generally be sufficiently accurate to use Comstock's formula, in the following form:

$$\text{Refraction in } \delta = -\frac{983b}{460+t} \tan z \cos q,$$

where z is the apparent zenith distance, and q the parallactic angle of the star, b the barometer reading in inches and t the temperature of the atmosphere in degrees Fahrenheit. In practice I have found it more convenient to correct each star for refraction in the manner described than to correct the difference in declination by the use of differential formulae.

The following pairs of stars in the *Pleiades* have actually been used by Professor Barnard in determining the value of one revolution of the micrometer screw of the forty-inch telescope of the Yerkes Observatory:

BD.	*Mag.*		*BD.*	*Mag.*	$\Delta\delta$
+23°537	(7.5)	and	+23°542	(8.2)	696.19″
+23°516	(4.8)	and	+23°513	(9.0)	285.94
+23°557	(4.0)	and	+23°559	(8.4)	599.58
+23°561	(7.5)	and	+23°562	(7.8)	479.11
+23°558	(6.2)	and	+23°562	(7.8)	401.10
+23°563	(7.2)	and	+23°569	(7.5)	494.14
+23°557	(4.0)	and	+23°558	(6.2)	300.25
+23°507	(4.7)	and	+23°505	(6.5)	633.40

The differences in declination given in the final column are for the epoch 1903.0 and are the results of Dr. Elkin's measures with the Yale heliometer. Dr. Trümpler has recently been making an accurate survey of the *Pleiades* cluster on the basis of photographs taken at the Allegheny Observatory, and when his results are published they may, to advantage, be combined with the values given above.

The last pair in the list consists of the bright stars *Electra* and *Celaeno*, and Professor Barnard kindly permits me to print in full his measures of them, made in 1912, to illustrate the use of step stars. The step stars in this case are respectively of magnitude 11.0 and 11.5 and lie nearly, but not quite, on the line joining the bright stars. Both the tube of the forty-inch telescope and the screw of the micrometer are of steel and therefore mutually correct each other in temperature changes, at least approximately; but the focal length of the object glass is three-fourths of an inch shorter in winter than in summer whereas the tube shortens only one-half of an inch. A slight correction is therefore necessary if all of the measures are to be reduced to the focus for a common temperature. The column *Scale reading*, in the table of measures, gives the

Measures of Δδ Electra and Celaeno

1912	Electra and *2	*2 and *1	*1 and Celaeno	Measured Δδ	H. A.	Refraction correction	Scale reading	Scale correction	True Δδ
Sept. 21	18^r.208	25^r.626	21^r.750	65^r.584	+2^h 30^m	+0^r.020	2.26	−0^r.004	65^r.600
22	18.195	25.601	21.752	65.548	+2 20	+0.020	2.30	−0.007	65.561
29	18.219	25.596	21.731	65.546	+2 25	+0.020	2.28	−0.006	65.560
Oct. 1	18.232	25.600	21.746	65.578	+1 40	+0.020	2.24	−0.003	65.595
6	18.224	25.604	21.726	65.554	+1 10	+0.019	2.28	−0.006	65.567
22	18.216	25.606	21.736	65.558	+2 0	+0.020	2.21	0.000	65.578
29	18.194	25.606	21.725	65.525	+2 40	+0.022	2.20	0.000	65.545
	18.213	25.606	21.738	65.556		+0.020		−0.003	65.572

This gives for one revolution of the screw:

$$\frac{633.44''}{65^r.572} = 9.66\dot{0}2''$$

Professor Barnard's measures from nine pairs of stars in 1902 and in 1912 lead to the mean value 9.6617″ at temperature 50° F.

readings for focus on the draw-tube of the telescope and the following column, the corrections required to reduce the measures to the focal length corresponding to a temperature of 50°F. The column *H. A.* gives the hour-angle at which the measures were made. The remaining columns are self-explanatory.

MEASURING A DOUBLE STAR

When the telescope has been directed upon the star and clamped, the star is brought up to the threads by means of the screw moving the entire micrometer box. The position angle is then measured, and in doing this my practice is to run the micrometer thread well to one side of the field of view, bring the double star up to the fixed thread by means of the screw moving the box and then rotate the micrometer by means of the pinion provided, keeping, meanwhile, the fixed thread upon the primary star, until the thread also passes centrally through the companion star. It is most convenient to manipulate the box screw with the left hand and the pinion with the right.

The tangent screw giving a slow motion in position angle is never used; in fact, it has been removed from the micrometer. When the seeing is good, the star images round, small, and steady, it is easy to hold both images on the thread until the eye is assured of their precise bisection. Under less favorable conditions a rapid to and fro motion of the box screw places the stars alternately on either side of the thread while the pinion is being rotated backward and forward until the eye is satisfied of the parallelism of the thread to the line joining the centers of the star images.

Ordinarily four independent settings for position angle are made, the circle being read, not by the vernier but by an index, directly to half degrees in the case of the twelve inch micrometer, to quarter degrees in the case of the thirty-six inch, and by estimation to the one-fifth of a division, *i. e.*, to 0.1° and 0.05°, respectively. After each setting the micrometer is rotated freely backward and forward, simply by turning the box directly with the hands, through an arc of 80° to 100°, the eye being removed from the eye-piece.

The circle is next set to a reading 90° greater (or less) than the mean of the readings for position angle and the distance is measured by bisecting one star with the fixed thread, the other with the micrometer thread. It is most convenient to turn the micrometer screw with the right hand, the box screw with the left. Then interchange the threads, placing the micrometer thread on the first star, the fixed thread on the other. The difference between the two readings of the micrometer screw-head gives the double-distance, *i. e.*, twice the angular separation, and eliminates the zero or coincidence reading. Three measures of the double distance are generally made. The milled head of the screw, which is divided to hundredths of a revolution, is read to the 1/1000 of a revolution by estimation. Care is always taken to run the micrometer thread back several seconds of arc after each setting and to make the final turn of the screw at each bisection *forward*, or against the springs.[2]

Any ordinary note book will answer as a record book. At the Lick Observatory, we have found convenient a book 7″ x 8½″ containing 150 pages of horizontally ruled, sized paper suitable for ink as well as pencil marks. The observing record is made in pencil, the reductions with ink. No printed forms are necessary or even convenient. A sample entry taken from my observing book shows the form of record adopted, and also the very simple reductions:

			36″ Sat. Jan. 27, 1917.
80 Tauri = Σ 554			Parallel = 10.25°
128.75° N. F. 0.9″ ±	49.401	.581	$4^h.3$
129.70 ΔM = 3	.400	.578	1000
129.30	.403	.580	2 to 2+
130.40	———	———	Well separated with
———	49.401	.580	520-power
129.54		.401	
100.25		———	
———		).179	
29.3° = θ_0		.089R = 0.88″ = ρ_0	

[2] The bisection of the star by the fixed thread should be made anew at each setting with the micrometer screw, because, under even the best conditions, it cannot be assumed that the star images will remain motionless during the time of observation.

Two such entries are ordinarily made to the page. The column at the left records the four settings for position angle; the mean is taken and the reading of the circle for parallel plus 90° is subtracted to obtain the position angle. Whether this value is the correct one or whether 180° is to be added to it is decided by the note made of the quadrant while observing —N. F. in the present case. When recording the quadrant, which is done after the position angle settings have been entered, I record also an *estimate* of the distance and of the difference of magnitude of the components. Sometimes, when the companion is very faint, I record, instead, a direct estimate of its magnitude. At this time, too, I record, at the right, the date, the sidereal time to the tenth of an hour, the power of the eye-piece used, an estimate of the seeing on a scale on which 5 stands for perfect conditions and any observing notes. Measures of distance are then made and recorded. Here the reduction consists in taking half the difference of the two means and multiplying the result by the value of one revolution of the micrometer screw (in this instance 9.907″).

The results are transfered to a "ledger", the date being recorded as a decimal of the year. The ledger entry for the above observation is:

80 Tauri = Σ554.

1917.075 29.3° 0.88″ $\Delta M = 3$ $4^h.3$ 1000 2 to 2 + bk. 87,147

the last item being the number and page of the observing book.

Practically all observers agree in the method of measuring the angular distance, but many prefer a somewhat different procedure for determining the position angle. They bring the two threads fairly close together—to a separation twice or three times the diameter of the primary's apparent disk—and then, placing the two stars between the threads, turn the micrometer until the line joining the stars appears to be parallel to the threads. I have found that I can secure equally satisfactory measures by this method when the two stars are well separated and of nearly equal magnitude, but not when the angular distance is small or when the stars differ much in brightness. While it may be a matter of personal adaptation I incline to think that measures made in this manner are more likely to be affected by systematic errors than those made by the method first described.

Whatever method is adopted it is of the first importance that the head of the observer be so held that the line between his eyes is parallel or perpendicular to the line joining the two stars. I can make the bisections with more assurance when the line between the eyes is parallel to the one joining the two stars, and hold my head accordingly unless the line is inclined more than 45° to the horizon. Some observers prefer the perpendicular position.

There are some other points that must be taken into consideration to secure satisfactory results. The star images as well as the threads must be brought sharply into focus; the images must be symmetrically placed with respect to the optical axis; and the threads must be uniformly illuminated on either side. In modern micrometers the illumination is usually provided by a small incandescent lamp placed in such a position that a small mirror can throw the light through a narrow opening in one end of the micrometer box. This mirror can be rotated through 90° thus permitting a variation in the intensity of the light from full illumination to zero. Suitable reflectors placed within the micrometer box, at the opposite end, insure equality in the illumination on both sides of the threads. Glass slides can also be placed in front of the opening admitting the light in order to vary its intensity or its color as may be desired. I have found no advantage in using these. The earlier double star observers frequently illuminated the field of view instead of the threads and an occasional observer still advocates this practice, but the great majority, I think, are agreed that this is a less satisfactory arrangement.

It is hardly necessary to say that the micrometer threads must be stretched to a tension sufficient to keep them perfectly straight even when the atmosphere is very moist and that they must be free from dust or other irregularities and accurately parallel. A cocoon of spider thread should be obtained from an instrument maker and kept on hand with the necessary adjusting tools and the micrometer threads replaced as often as they become unsatisfactory. A little practice will enable the observer to set a thread in position in a very short space of time; in fact, from Burnham's days to the present time, a new thread has frequently been set into the thirty-six-inch micrometer during the night and observing been resumed within an hour.

The most important precaution to be taken in double star observing is quite independent of instrumental adjustments. It is *to make measures only on nights when the observing conditions are good.* Measures made under poor observing conditions are at best of little value, and at worst are a positive hindrance to the student of double star motions. They annoy or mislead him in his preliminary investigations and are practically rejected in his later work. I make this statement with all possible emphasis.

It is of almost equal consequence to select stars suited to the power of the telescope employed. This, however, is to a considerable extent a matter involving the personal equation. A Dawes, a Dembowski, or a Burnham can measure with comparatively small apertures stars that other observers find difficult with much larger telescopes.

MAGNITUDE ESTIMATES

It is well known that the magnitudes assigned to the components of the same double star by different observers frequently show a range that is excessively large. Whatever excuse there may have been for this in earlier days, there is certainly little at the present time when the magnitudes of all of the brighter stars are given in the photometric catalogues and those of all stars to at least 9.5 magnitude in the various *Durchmusterungs.* It is certainly advisable to take the combined magnitude of the two components (or the magnitude of the brighter star, if the companion is very faint) from these sources instead of making entirely independent estimates. The *difference of magnitude* is then the only quantity the double star observer need estimate. If this difference is not too great it can be estimated with comparative accuracy; if one component is very faint, a direct estimate of its brightness may be based upon the limiting magnitude visible in the telescope used, care being taken to allow for the effect of the bright companion which will always make the faint star appear fainter than it really is.

To derive the brightness of each component when the combined magnitude and the difference of magnitude are known, we have the relations, A = C + x, B = A + d, in which A and B are the magnitudes of the brighter and fainter component, respectively, C the combined magnitude, and d the estimated difference of magnitude, while x is given by the equation

$$x = \frac{\log\left(1 + \frac{1}{2.512^d}\right)}{0.4}$$

We may tabulate x for different values of d as in the following table which is abbreviated from the one in Innes's Reference Catalogue:

d	x	d	x
0.0	0.75	1.5	0.25
0.25	0.6	2.0	0.15
0.5	0.5	2.5	0.1
0.75	0.4	3.0	0.05
1.0	0.3	4.0	0.0

To illustrate the use of the table let d, the observed difference in brightness, be three-fourths magnitude (it is sufficiently accurate to estimate the difference to the nearest quarter magnitude), and let the photometric magnitude, C, be 7.0. Then, from the table, x = 0.4, and the magnitudes of A and B are 7.4 and 8.2 (to the nearest even tenth). Conversely, we may find C from A and B.

THE OBSERVING PROGRAM

It has happened in the past that certain well-known double stars have been measured and remeasured beyond all reasonable need, while other systems of equal importance have been almost entirely neglected. The General Catalogues described in the preceding chapter make it comparatively easy for observers to avoid such mistakes hereafter. In the light of the knowledge these catalogues give of past observations and of the motions in the various systems, the observer who wishes his work to be of the greatest possible value will select stars which are suited to his telescope and which are in need of measurement at a given epoch either because of scarcity of earlier measures or because the companion is at a critical point in its orbit. For example, I am measuring β 395 = 82 *Ceti* regularly at present because there are no measures of the companion in the part of the orbit through which it is now moving, and I am watching ϵ *Equulei* closely because at present it is apparently single and the position of the companion at the time of its reappearance will practically decide the character of the orbit.

It has often been said that a careful set of measures of any pair of stars made at any time is valuable. Granting this to be so, it is certain that its value is greatly enhanced if it is made to contribute to the advancement of a program having a definite end in view. If the aim is to increase the number of known orbits as rapidly as possible, attention should be centered upon the closer pairs, particularly those under 0.5″ and those which have already been observed over considerable arcs of their orbits. I am emphatically of the opinion that it is wise for an observer possessing the necessary telescopic equipment to devote his energy to the measurement of a limited number of such systems, repeating the measures every year, or every two or five years, as may be required by the rapidity of the orbital motion, for a long series of years. Such a series can be investigated for systematic as well as accidental errors of measure far more effectively than an equal number of measures scattered over a much larger program, and will add more to our real knowledge of the binary systems. The wider pairs, and particularly those in the older catalogues, now need comparatively little attention, so far as orbital motion is concerned. Even moderately close pairs, with distance from 1″ to 5″, need, in general, to be measured but once in every ten or twenty years. Useful programs, however, may be made from wider pairs for the detection of proper motions, or for the determination of the relative masses in binary systems by means of measures connecting one of the components with one or more distant independent stars. Photography is well adapted to such programs.

It is hardly necessary to add that an hour in the dome on a good night is more valuable than half a dozen hours at the desk in daylight. Everything possible should therefore be done to prevent loss of observing time. In this connection I have found charts based on the *Durchmusterung* invaluable for quick identification of stars.

THE RESOLVING POWER OF A TELESCOPE

It has been shown that the diffraction pattern of the image of a point source of light, like a star, formed by a lens "is a disk surrounded by bright rings, which are separated by circles at which the intensity vanishes."[3]

Schuster gives the formula

$$\rho = m \frac{fx}{D} \qquad (5)$$

in which ρ is the radius of a circle of zero intensity (dark ring), D the diameter of the lens, f its focal length, λ the wave length of the light from the point source, and m a coefficient that must be calculated for each ring. For the first dark ring it is 1.220, and the values for the successive rings increase by very nearly one unit. Nearly all of the light (0.839) is in the central disk, and the intensity of the bright diffraction rings falls off very rapidly. Now it is generally agreed that the minimum distance at which a double star can be distinctly seen as two separate stars is reached when the central disk of the image of the companion star falls upon the first dark ring of the image of the primary, and the radius of this ring, expressed in seconds of arc, is therefore frequently called the limit of the telescope's resolving power. If we adopt for λ the wave length 5,500 Ångström units, the expression for ρ in angular measure becomes

$$\rho = \frac{5.03''}{D} \qquad (6)$$

[3] Schuster, *Theory of Optics* (1904), p. 130.

from which the resolving power of a telescope of aperture D (in inches) may be obtained. For the thirty-six-inch Lick refractor, the formula gives 0.14″, for the twelve-inch, 0.42″.

It will be observed that the resolving power as thus derived rests partly upon a theoretical and partly upon an empirical basis. When the central disk of each star image of a pair falls upon the first dark ring of the other image, the intensity curve of the combined image will show two maxima separated by a distinct minimum. When the disks fall closer together this minimum disappears, the image becomes merely elongated, perhaps with slight notches to mark the position of the disappearing minimum. The pair is now no longer 'resolved' according to the definition given but to the experienced observer its character may still be unmistakable. For example, in the Lick Observatory double star survey, Hussey and I have found with the thirty-six-inch at least 5 double stars with measured distances of 0.11″ or less, the minimum for each observer being 0.09″; and we have found many pairs with the twelve-inch telescope whose distances, measured afterward with the thirty-six-inch, range from 0.20″ to 0.25″. In all these cases the magnitudes were, of course, nearly equal.

Lewis has published [4] a very interesting table of the most difficult double stars measured and discovered by various observers using telescopes ranging in aperture from four inches to thirty-six inches. He tabulates in separate columns the values for the 'bright' and 'faint' pairs of nearly equal magnitude, and for the bright and faint pairs of unequal magnitude, each value representing the mean of 'about five' of the closest pairs for a given observer and telescope. A final column adds the theoretical resolving power derived, not from the equation given above, but from Dawes's well-known empirical formula—resolving power equal 4.56″ divided by the aperture in inches (a)—which assumes the two stars to be of about the sixth magnitude. Lewis finds that, in general, this formula gives values which are too small even for the 'bright equal

[4] The *Observatory*, vol. xxxvii, p. 378, 1914.

pairs', and he suggests the following as representing more precisely the results of observation:

Equal bright pairs	$\frac{4.8''}{a}$,	mean magnitudes 5.7 and 6.4
Equal faint pairs	$\frac{8.5''}{a}$,	mean magnitudes 8.5 and 9.1
Unequal pairs	$\frac{16.5''}{a}$,	mean magnitudes 6.2 and 9.5
Very unequal pairs	$\frac{36.0''}{a}$,	mean magnitudes 4.7 and 10.4

Lewis is careful to state that his table does not necessarily represent the minimum limits that may be reached with a given telescope under the best conditions, and I have just shown that they do not represent the limits actually reached at the Lick Observatory. Taking from each of the three lists of new double stars β 1,026 to β 1,274, Hu 1 to Hu 1,327, and A 1 to A 2,700, 'about five' of the closest bright, and closest faint, equal pairs discovered by each of the three observers, Burnham, Hussey, and Aitken—29 pairs in all—I find the following formulae for the thirty-six-inch telescope:

Equal bright pairs	$\frac{4.3''}{a}$,	mean magnitudes 6.9 and 7.1
Equal faint pairs	$\frac{6.1''}{a}$,	mean magnitudes 8.8 and 9.0

The most interesting point about these formulae is that they show much less difference between the values for faint and bright pairs than Lewis's do.

While it is a matter of decided interest to compare the limits actually attained with a given telescope with the theoretical resolving power, an observer, in making out his working program for double star measurement, will do well to select pairs that run considerably above such limiting distances. My deliberate judgment is that, under *average good observing conditions*, the angular separation of the pairs measured should be nearly double the theoretical limit. Observers with the most powerful telescopes, however, are confronted with the fact that if they do not measure the very closest known pairs these must go unmeasured.

EYE-PIECES

The power of the eye-piece to be used is a matter of practical importance, but one for which it is not easy to lay down specific rules. The general principle is—*use the highest power the seeing will permit.* When the seeing is poor, the images 'dancing' or 'blurred', increase in the magnifying power increases these defects in the images and frequently more than offsets in this way the gain from increase in the scale. On such nights, if they are suitable for any work, choose wider pairs and use lower powers. The practical observer soon realizes that it is not worth while to measure close pairs except with high powers. With the thirty-six-inch telescope my own practice is to use an eye-piece magnifying about 520 diameters for pairs with angular separation of 2″ or more. If the distance is only 1″, I prefer a power of 1,000, and for pairs under 0.5″, I use powers from 1,000 to 3,000, according to the angular distance and the conditions. Closeness and brightness of the pair and the quality of the definition are the factors that determine the choice. Very close pairs are never attempted unless powers of 1,500 or higher can be used to advantage.

The simplest method of measuring the magnifying power of an eye-piece in conjunction with a given objective is to find the ratio of the diameter of the objective to that of its image formed by the eye-piece—the telescope being focused and directed to the bright daylight sky. Two fine lines ruled on a piece of oiled paper to open at a small angle form a convenient gage for measuring the diameter of the image. A very small error in this measure, however, produces a large error in the ratio and the measure should be repeated many times and the mean result adopted.

DIAPHRAGMS

It is sometimes said that the quality of star images is improved by placing a diaphragm over the objective to cut down its aperture. I question this. It is certain that the experience of such observers as Schiaparelli and Burnham is directly opposed to it, and experiments made with the twelve-inch and thirty-six-inch telescopes offer no support for it. Indeed, it is difficult to understand how cutting off part of the beam of light falling upon an object glass of good figure can improve the character of the image, unless it is assumed that the amplitude of such atmospheric disturbances as affect the definition is small enough to enter the problem. The only possible gain might be in the reduction of the brightness of the image when one star of a pair is exceptionally bright, as in *Sirius;* but this reduction can be effected more conveniently by the use of colored shade glasses over the eye-piece. These are occasionally of advantage.

A hexagonal diaphragm placed over the objective, however, may prove of great value in measuring stars, like *Sirius* or *Procyon*, which are attended by companions relatively very faint; but this is because such a diaphragm entirely changes the pattern of the diffraction image of the star, not because it cuts down the aperture of the telescope. The pattern is now a central disk from which six thin rays run; between these rays the field appears dark even close to the bright star, and a faint object there can be seen readily that would be invisible otherwise. Professor Barnard [5] has used such a diaphragm to advantage with the forty-inch Yerkes refractor. Provision should be made for rotating the diaphragm through an angle of about 60° and it will be convenient in the case of a large instrument to be able to do this by means of gearing attached to a rod running down to the eye-end.

ERRORS OF OBSERVATION

All measures of angles or of distances are affected by errors, both accidental and systematic, and when, as in double star work, the measured quantities are very minute, these errors must be most carefully considered. The accidental errors may be reduced by careful work and by repeating the measures a suitable number of times. Little is to be gained, in this respect or in any other, by making too large a number of set-

[5] *A. N.*, vol. clxxxii, p. 13, 1909.

tings upon an object on any one night; because such factors as the seeing, the hour angle, the observer's physiological condition, all remain nearly constant. As a rule, four settings for position angle and three or even two measures of double distance are enough to make on one night, but the measures should be repeated on one or more additional nights. This is not only to reduce the accidental error of measure but to guard against outright mistakes in reading the circles, recording, etc. As to the number of nights on which a system should be measured at a given epoch, opinions will differ. Some observers run to excess in this matter. Generally, it may be said that it is time wasted to measure a system on more than four nights at any epoch and ordinarily the mean of three nights' measures, or even of two, if the pair is easy to measure and the measures themselves are accordant, is as satisfactory as the mean from a larger number. In critical cases, however, a larger number is sometimes desirable.

The systematic errors of measurement are far more troublesome, for they vary not only with the individual but are different for the same observer at different times and for different objects. Aside from the personality of the observer, they depend upon the relative magnitudes of the two components of a double star, the angular distance, the angle which the line joining the stars makes with the horizontal, and, in unequal pairs, upon the position of the faint star with respect to the bright one. Various methods have been adopted to determine these errors or to eliminate them.

The most elaborate investigation in this line is probably that made by Otto Struve, who measured "artificial double stars formed by small ivory cylinders placed in holes in a black disk." He deduced formulae by means of which he calculated corrections to be applied to all of his measures; but it is very doubtful whether these corrections really improve the results. I agree with Lewis when he says, "I would prefer his original measures—in part because the stars were so particularly artificial." The actual conditions when observing the stars at night are of necessity widely different from those under which the test measures were made. Certainly, in the case of Otto

Struve, the 'corrected' angles and distances are frequently more at variance with the general run of all of the measures by good observers than the original values. The student of double star motions will generally find it advantageous to use the original uncorrected measures of every observer in his preliminary work and then to derive values for the systematic or personal errors of each by comparing his measures with the curve representing the means of all available measures.

The observer, on the other hand, may profitably adopt observing methods designed to eliminate, in part at least, systematic errors. Innes's plan of measuring each pair on each side of the meridian is an excellent one because, in general, the line joining the two stars changes its angle with respect to the horizon in passing the meridian. In the extreme case, if the smaller star is above the primary when the pair is east of the meridian, it will be below, when west of the meridian. When Innes's two measures made in this way are not sufficiently accordant, he repeats them on two additional nights, one night in each position of the instrument.

In 1908, MM. Salet and Bosler [6] published the results of an investigation of the systematic errors in measures of position angle in which they made use of a small total reflecting prism mounted between the eye-piece and the observer's eye and capable of being rotated in such manner as to invert the field of view. Theoretically, the half sum of the measures made without and with the prism should represent the angle freed from errors depending upon the inclination of the images to the horizon. In fact, Salet and Bosler found that, whereas their measures without the prism and those made with it both showed a personal equation varying in amount with the star, the means of the two sets were remarkably free from personality. Here, for example, are their measures of γ *Leonis:*

Observer	Date	Without Prism	With Prism	Mean
Salet	1907.19	119.04°	113.50°	116.27°
Bosler	1907.23	116.80°	116.07°	116.44°
(S–B)		+2.24°	—2.57°	−0.17°

[6] *Bulletin Astronomique*, Tome xxv, p. 18, 1908.

Hermann Struve and J. Voûte have since published measures made in this manner and each concludes that the results are far better than his meaures made entirely without the use of the prism. In M. Voûte's last paper[7] the statement is made that "it is principally in observing in the perpendicular (:) position that the observations show a pronounced systematic error," while "the parallel (. .) observations are in general free from systematic errors."

Dawes[8] long ago pointed out that in "rather close double stars," the measures of distance "will almost inevitably be considerably *too large*," unless the observer has taken into account the change made in the apparent form of the star disk when a thread of the micrometer is placed over it. This change is in the nature of a swelling out of the disk on each side of the thread, producing an approximately elliptical disk. When two images are nearly in contact and the threads are placed over them this swelling obliterates the interval between the disks and the threads are therefore set *too far apart.* The effect disappears when the disks are well separated.

In my investigations of double star orbits I have frequently noticed that distance measures of a given system made with small apertures are apt to be greater than those made with large telescopes even when made by the same observer, provided the system is a close one as viewed in the smaller instrument. I have found such a systematic difference in the distances in stars which I have measured with the twelve-inch and with the thirty-six-inch telescope, and Schlesinger[9] has also called attention to this difference, giving a table derived from my measures as printed in Volume XII of the *Publications of the Lick Observatory.* This table is here reproduced with a column of differences added:

[7] *Circular No. 27, Union Observatory*, South Africa, 1915.
[8] *Mem. Royal Astronomical Society*, vol. xxxv, p. 153, 1867.
[9] *Science*, N. S., vol. xliv, p. 573, 1916.

Measured Separations

Number of Stars	With the 12-inch	With the 36-inch	Difference
20	0.52″	0.42″	+0.10″
25	0.62″	0.54″	+0.08″
20	0.71″	0.64″	+0.07″
24	0.81″	0.79″	+0.02″
24	1.07″	1.03″	+0.04″
21	1.38″	1.39″	—0.01″
26	2.13″	2.10″	+0.03″
18	4.49″	4.53″	—0.04″

The systematic difference is clearly shown in all the pairs having a separation less than twice the resolving power (0.42″) of the twelve-inch telescope; in the wider pairs it is negligibly small.

Occasionally an observer's work shows systematic differences of precisely the opposite sign. Thus Schlesinger (*l. c.*) shows that in Fox's recent volume[10] of double star observations the distances are measured smaller with the twelve-inch than with the eighteen and one-half-inch or with the forty-inch, "the differences being largest for small separations and becoming negligibly small for separations in the neighborhood of 5″." The personal equation revealed in such comparisons as these must obviously be taken into account in orbit computations.

PHOTOGRAPHIC MEASURES

Photographic processes of measurement are coming more and more into favor in almost all lines of astronomical work, and with the constant improvements that are being made in the sensitiveness and fineness of grain in the plates it is probable that important work in double star measurement will soon be undertaken photographically. Indeed, experiments in this line date back to 1857, when G. P. Bond secured with an eight-second exposure on a collodion plate the first measurable images of a double star—ζ *Ursae Majoris*, angular

[10] *Annals of the Dearborn Observatory, Northwestern University*, vol. i, 1915.

separation 14.2″. Pickering and Gould in America, MM. Henry in France, and the Greenwich observers in England, among others, followed up this early attempt and succeeded in securing results of value for some stars as close as 1″. More recently Thiele, Lau, and Hertzsprung at Copenhagen have carried out more extensive programs and have investigated several possible sources of systematic error in the measurement of photographic plates. Hertzsprung is at present continuing his researches at Potsdam, and finds it possible to secure excellent measures of pairs as close as 1″.

There are obvious limitations to the application of photography to double star measurement; very close pairs and pairs with moderate distances in which one component is relatively faint will not give measurable images on plates at present available. On the other hand there is no question but that wider pairs can be as accurately, and far more conveniently measured photographically than visually, provided systematic errors are eliminated. The discovery of faint double stars with distances exceeding, say 3″, may also with advantage be left to the photographic observer. Comparison under the blink microscope of plates taken at suitable intervals with instruments giving fields on the scale of the *Carte-du-ciel* charts will quickly reveal any such pairs which show appreciable motion and these are the only faint pairs that need be taken into serious account in the present stage of double star astronomy. Instruments giving photographs of larger scale will, of course, reveal closer pairs. Thus, Fox, on plates exposed with the eighteen and one-half-inch Dearborn refractor, has recently found two pairs (Fox 11 and Fox 25) with measured distances of 1.7″ and 1.2″, the magnitudes being 9.9—11.0 and 9.6—10.2 respectively.

CHAPTER IV

THE ORBIT OF A VISUAL BINARY STAR

We have seen that Sir William Herschel, by his analysis of the observed motion in *Castor* and other double stars, demonstrated that these systems were "real binary combinations of two stars, intimately held together by the bonds of mutual attraction." Later observation has shown that the apparent motion in such systems is on the arc of an ellipse and that the radius vector drawn from the primary star to its companion sweeps over areas which are proportional to the times. It has therefore been assumed from the beginning that the attractive force in the binary star systems is identical with the force of gravitation in our solar system, as expressed by Newton's law, and the orbit theories which we are to investigate in the present chapter are all based upon this assumption. Before taking up the discussion of these theories it is pertinent to inquire whether the fundamental assumption is justified.

It is supported by all of the available evidence but rigorous mathematical proof of its validity is difficult because the motion which we observe in a stellar system is not the true motion but its projection upon a plane perpendicular to the line of sight. The apparent orbit is therefore, in general, not identical with the true orbit and the principal star may lie at any point within the ellipse described by the companion and not necessarily at either the focus or the center. Hence, in Leuschner's words, "mathematical difficulties are encountered in establishing a law of force which is independent of the angle θ, the orientation." In the article quoted, Leuschner, after pointing out that "Newton did not prove the universality of the law of gravitation, but by a happy stroke of genius generalized a fact which he had found to be true in the case of the mutual attraction of the Moon and the Earth," proceeds to show that the law does hold throughout the solar system, the question of orientation not entering. He then says that, in binary systems, "when the law is arbitrarily assumed to be independent of the orientation, as was found to be the case in the solar system, two possibilities arise, namely, either that the force is in direct proportion to the distance r between the two stars or that the Newtonian law applies. It can be shown, however, that when, in the case of an elliptic orbit, the force is proportional to r, the primary star must be in the center of the ellipse. As this has never been found to be the case, the only alternative is the Newtonian law."

It should be clearly understood that the difficulty in demonstrating the universality of the law of gravitation here pointed out is purely mathematical. No physical reason has ever been advanced for a dependence of an attracting central force upon the orientation, and until such dependence has been proved we may safely proceed with our investigation of binary star orbits under the action of the law of gravitation.

Until the relative masses of the two components are known it is impossible to determine the position of the center of gravity of the system and we are therefore unable to compute the orbits described by the two stars about that center. What our measures give us is the apparent orbit of one star, the companion, described about the other, the primary, which is assumed to remain stationary at the focus. It is clear that this relative orbit differs from the actual orbits of the two components only in its scale.

The problem of deriving such an orbit from the micrometer measures of position angle and distance was first solved by Savary,[1] in 1827, but Encke [1] quickly followed with a different method of solution which was somewhat better adapted to the needs of the practical astronomer, and Sir John Herschel [1] communicated a third method to the Royal Astronomical Society in 1832. Since then the contributions to the subject have been many. Some consist of entirely new methods of attack, others of modifications of those already proposed. Among the more notable investigators are Villarceau, Mädler,

[1] Savary, *Conn. des Temps*, 1830; Encke, *Berlin Jahrbuch*, 1832; Herschel, *Memoirs of the Royal Astronomical Society*, 5, 171, 1833.

Klinkerfues, Thiele, Kowalsky, Glasenapp, Seeliger, Zwiers, Howard, Schwarzschild, See, and Russell.

The methods of Savary and Encke utilize four complete measures of angle and distance and, theoretically, are excellent solutions of the problem; Herschel's method is designed to utilize all the available data, so far as he considered it reliable. This idea has commended itself to all later investigators. Herschel was convinced, however, that the measures of distance were far less trustworthy than those of position angle, and his method therefore uses the measures of distance simply to define the semi-major axis of the orbit; all of the other elements depend upon measures of position angle. At the time this may have been the wisest course, but the distance measures of such early observers as W. Struve, Dawes, and Dembowski, and those of later observers working with modern micrometers, are entitled to nearly or quite as much weight as the measures of position angle and should be utilized in the entire orbit computation.

Whatever method is adopted it is clear that the problem consists of two distinct parts: *first*, the determination of the apparent ellipse from the data of observation; *secondly*, the derivation of the true orbit by means of the relations between an ellipse and its orthographic projection.

THE APPARENT ELLIPSE

Every complete observation of a double star supplies us with three data: the time of observation, the position angle of the companion with respect to the primary, and the angular distance between the two stars. It is clear, as Comstock pointed out many years ago, that the time of observation is known with far greater accuracy than either of the two coordinates of position. The relations between the times of observation and the motion in the ellipse should therefore be utilized; that is, the condition should be imposed that the law of areal velocity must be satisfied as well as the condition that the points of observation should fall approximately upon the curve of an ellipse. Elementary as this direction is, it is one that has been neglected in many a computation.

Theoretically, the first step in our computation should be the reduction of the measured coordinates to a common epoch by the application to the position angles of corrections for precession and for the proper motion of the system. The distance measures need no corrections. Practically, both corrections are negligibly small unless the star is near the Pole, its proper motion unusually large, and the time covered by the observations long. The precession correction, when required, can be found with sufficient accuracy from the approximate formula

$$\Delta\theta = -0.0056° \sin \alpha \sec \delta \, (t - t_0). \qquad (1)$$

The formula for the correction due to the proper motion of the system is

$$\Delta\theta = \mu'' \sin \delta \, (t - t_0) \qquad (2)$$

where μ'' is the proper motion in right ascension expressed in seconds of arc.

When the measures of any binary star have been tabulated (with the above corrections, if required) they will exhibit discordances due to the accidental and systematic errors of observation and, occasionally, to actual mistakes. If they were plotted, the points would not fall upon an ellipse but would be joined by a very irregular broken line indicating an ellipse only in a general way. It will be advisable to investigate the measures for discordances before using them in the construction of the apparent ellipse and the simplest method is to plot upon coordinate paper first the position angles and then the distances, separately, as ordinates, against the times of observation as abscissae, using a fairly large scale. Well determined points (for example, a point resting upon several accordant measures by a skilled observer and supported by the preceding and following observations) may be indicated by heavier marks. Smooth free-hand curves, interpolating curves, are now to be drawn to represent the general run of the measures and in drawing these curves more consideration will naturally be given to the well observed points than to the others. Observations which are seriously in error will be clearly revealed and these should be rejected if no means of correcting them is available. The curves will also show whether or not the measures as a whole are sufficiently good to make orbit computation desirable.

If the amount of available material warrants it, the question of the systematic or personal errors of the observers should also be considered at this time. No reliable determination of such errors is possible unless (*a*) measures by the same observer under essentially the same conditions in at least four or five different years are at hand, and (*b*) unless the total number of measures by many different observers is sufficient to establish the general character of the curves beyond reasonable question. If the second condition is satisfied, the average of the residuals from the curve for a given observer may be regarded as his personal error and the corresponding correction may be applied to all of his measures. Two further points should be noted: *First*, the residuals in position angle should be reduced to arc by multiplying by the factor $\rho \div 57.3$ before the mean is taken, to allow for the effect of variations in the angular separation; *second*, the corrections should not be considered as constant over too long a period of time. Many computers take no account of the personal errors in their calculations, and if the object is merely to obtain a rough preliminary orbit this practice is perhaps legitimate.

After all corrections have been applied, the measures which are retained should be combined into annual means or into mean places at longer or shorter time intervals according to the requirements of the particular case. Several factors really enter into the question of the weights to be assigned to the individual observations in forming these means; for instance, the size of the telescope used, the observing conditions, the number of nights of observation, and the experience of the observer; but it will be wise, in general, to disregard all but the number of nights of observation, provided the telescope used is of adequate resolving power for the system in question and that the observer has not specifically noted some of his measures as uncertain. A single night's measure deserves small weight; mean results based upon from two to six nights' accordant measures may be regarded as of equal weight; means depending upon a much larger number of measures may be weighted higher. In general, a range in weights from one to three will be sufficient.

Having thus formed a series of normal places, we may find the apparent ellipse which best represents them either graphically or by calculating the constants of the general equation of the ellipse with the origin at any point. This equation is

$$ax^2+2hxy+by^2+2gx+2fy+c=0 \qquad (3)$$

which may be written in the form

$$Ax^2+2Hxy+By^2+2Gx+2Fy+1=0 \qquad (4)$$

in which we must have $A>0$, $B>0$, and $AB-H^2>0$.

If we assume the position of the primary star as origin we may calculate the five constants of this equation from as many normal places by the relations

$$\begin{aligned} x&=\rho \sin \theta \\ y&=\rho \cos \theta \end{aligned} \qquad (5)$$

but it is advisable to make a least squares solution using all of the normal places.

The great objection to this method is that it entirely disregards the times of observation. Moreover, the errors of observation, small as they are numerically, are large in proportion to the quantities to be measured, a fact which makes it difficult to obtain a satisfactory ellipse without repeated trials. The graphical methods are therefore to be preferred.

The simplest method, and one that in most cases is satisfactory, is to plot the positions of the companion star in polar coordinates, the primary star being taken as the origin. With the aid of an ellipsograph or by the use of two pins and a thread, an ellipse is drawn through the plotted points and is adjusted by trial until it satisfies the law of areas. *This adjustment must be made with the greatest precision and the curve of the ellipse drawn with great care*, for the construction of the apparent ellipse is the critical part of the entire orbit determination. In my own practice I have found that the test for the law of areas can be made most rapidly by drawing radii to selected points which cover the entire observed arc and measuring the corresponding elliptic sectors with a planimeter. The comparison of the areal velocities derived from the different sectors at once indicates what corrections the ellipse requires. With a suitable ellipsograph a new ellipse is quickly drawn and the areas again measured. The process is repeated until a satisfactory ellipse has been obtained.

Some investigators still prefer the mode of procedure in constructing the apparent ellipse first suggested by Sir John Herschel. An interpolating curve is drawn, in the manner described above, for the *position angles only*, using the mean or normal places. If the curve is carefully drawn, smoothly and without abrupt changes of curvature, it should give the position angle for any particular epoch more accurately than the measure at that epoch, for it rests upon all of the measures. From this curve read the times corresponding to, say, every 5° of angle, tabulate them, and take the first differences. Dividing these by the common angle difference will give a series of approximate values of $\frac{dt}{d\theta}$. But by the theory of elliptic motion $\rho^2 \frac{d\theta}{dt}$ must be a constant and hence $\rho = c\sqrt{\frac{dt}{d\theta}}$. Therefore a series of *relative* values of the distance (expressed in any convenient unit) corresponding to every fifth degree of position angle can be derived from the table of angles. Now plot the points representing corresponding angles and relative distances; if the interpolating curve has been correctly drawn and read off they will all lie upon the arc of an ellipse. If they do not, draw the best possible ellipse among them and use it to correct the interpolating curve, repeating the process until the result is satisfactory. Finally, convert the relative into true distances by comparing those distance measures which are regarded as most reliable with the corresponding values in the unit adopted in the plot.

There are at least two objections to this method: *First*, it does not make adequate use of the observed distances; and *second*, when the angle changes rapidly, as it does in many systems at the time of minimum apparent separation, it is almost impossible to draw the interpolating curve correctly.

In my judgment, it is far better to plot directly the normal positions given by the observed angles and distances and then by the method of trial and error to find the ellipse which best represents them and at the same time satisfies the law of areas.

THE TRUE ORBIT

After the apparent ellipse has been constructed graphically, or from the constants in the equation of the ellipse, it remains to derive the elements which define the form and size of the true orbit, the position of the orbit plane, the position of the orbit within that plane, and the position of the companion star in the orbit at any specified time. Some confusion in the nomenclature and even in the systems used in defining these elements has arisen from the fact that it is impossible to say, from the micrometer measures alone, on which side of the plane of projection (which is taken as the plane of reference) the companion star lies at a given time. In other words, we cannot distinguish between the ascending and the descending node, nor between direct and retrograde motion in the ordinary sense. Further, in some systems the observed position angles *increase* with the times, in others they *decrease*.

The following system is adopted as most convenient when the requirements of the observer of radial velocities are considered as well as those of the observer with the micrometer. In the details in which it differs from other systems in use, it was worked out by Dr. Campbell in consultation with Professor Hussey and the present writer.

Let

P = the period of revolution expressed in mean solar years.

T = the time of periastron passage.

e = the eccentricity.

a = the semi-axis major expressed in seconds of arc.

Ω = the position angle of that nodal point which lies between 0° and 180°; that is, the position angle of the line of intersection of the orbit plane with the plane perpendicular to the line of sight. Call this merely "the nodal point", disregarding the distinction between ascending and descending nodes.

ω = the angle in the plane of the true orbit between the line of nodes and the major axis. It is to be measured from the nodal point to the point of periastron passage in the direction of the companion's motion and may have any value from 0° to 360°. It should be stated whether the position angles increase or decrease with the times.

i = the inclination of the orbit plane; that is, the angle between the orbit plane and the plane at right angles to the line of sight. Its value lies between 0° and ±90° and should always carry the double sign (±) until the indetermination has been removed by measures of the radial velocity. When these are available, i is to be regarded as positive (+) if the orbital motion at the nodal point is carrying the companion star away from the observer; negative, if it is carrying the companion star towards the observer.

The symbol μ denotes the mean annual motion of the companion, expressed in degrees and decimals, *measured always in the direction of motion.*

There is no difference of opinion in regard to the definition of the first four elements; the conventions of taking Ω always less than 180° and of counting ω (for which many computers use the symbol λ) always in the direction of the companion's motion were first suggested, I believe, by See, and have been adopted by Burnham, Hussey, Aitken and others. The definition of i (for which some computers write γ) is the usual one, but computers, as a rule, do not use the double sign. Many also prefer to count the mean annual motion in the direction of increasing position angles in all systems, and to consider the motion negative when the angles decrease with the times.

When the elements are known, the apparent position angle θ and the angular distance ρ for the time t are derived from the following equations:

$$\begin{aligned}
\mu &= 360^\circ / P \\
M &= \mu(t-T) = E - e \sin E \\
r &= a(1 - e \cos E) \qquad (6) \\
\tan \tfrac{1}{2} v &= \sqrt{\frac{1+e}{1-e}} \tan \tfrac{1}{2} E
\end{aligned}$$

$$\left.\begin{aligned}
\tan(\theta - \Omega) &= \pm \tan(v+\omega) \cos i \\
\rho &= r \cos(v+\omega) \sec(\theta - \Omega)
\end{aligned}\right\} \quad (7)$$

Equations (6) are the usual ones for elliptic motion, the symbols M, E, and v representing respectively, the mean, eccentric, and true anomaly, and r the radius vector. Equations (7) convert the v and r of the companion in the true orbit into its position angle and distance in the projected, or apparent orbit. Position angles are generally recorded only to the nearest tenth of a degree in orbit computation, hence it is sufficiently exact to take the value of E corresponding to a given value of M from Astrand's Hülfstafeln, which hold for all values of the eccentricity, or the value of v directly from the still more convenient Allegheny Tables, provided the eccentricity does not exceed 0.77. If the latter tables are used it is convenient to derive the value of r from the equation

$$r = a\,(1 - e^2)\;/\;(1 + e\cos v) \qquad \text{(6a)}$$

instead of from the third of (6).

KOWALSKY'S METHOD

From the many methods which have been devised for deriving the elements of the true orbit from the apparent ellipse I have selected two to present in detail, Kowalsky's and Zwiers's. Both are of very general application and are very convenient in practice but there are cases in which both fail. Some of these will be discussed on a later page.

Kowalsky's method [2] is essentially analytical and employs the constants of the general equation of the ellipse.

$$Ax^2 + 2\,Hxy + By^2 + 2\,Gx + 2\,Fy + 1 = 0. \qquad (4)$$

This is the equation for the rectangular projection of the true orbit, the focus of the true orbit falling upon the position of the principal star at the origin of coordinates for the apparent ellipse. Equation (4) may also be regarded as the equation of a right cylinder whose axis coincides with the z-axis of the system of coordinates, that is, the line of sight. Let this equation be referred to a new system of coordinates, x', y', z', with the same origin, but with the x'-axis directed

[2] First published, according to Glasenapp, in the Proceedings of the Kasan Imperial University, 1873. This volume has not been accessible to me.

to the nodal point, and the y'-axis at right angles to it in the plane of the true orbit. Our transformation equations are

$$\begin{aligned} x &= x' \cos\Omega - y' \sin\Omega \cos i + z' \sin\Omega \sin i \\ y &= x' \sin\Omega + y' \cos\Omega \cos i - z' \cos\Omega \sin i \quad (8) \\ z &= \qquad\quad\; + y' \sin i \qquad\quad\;\; + z' \cos i \end{aligned}$$

Substituting these values in (4) and placing $z' = 0$, we shall have the equation of the intersection of the cylinder with the plane of the true orbit; that is, the real ellipse. Omitting accents the equation now becomes

$$\begin{aligned} &A(x\cos\Omega - y\sin\Omega\cos i)^2 \\ &\quad + 2H(x\cos\Omega - y\sin\Omega\cos i)(x\sin\Omega + y\cos\Omega\cos i) \\ &+ B(x\sin\Omega + y\cos\Omega\cos i)^2 + 2G(x\cos\Omega - y\sin\Omega\cos i) \\ &+ 2F(x\sin\Omega + y\cos\Omega\cos i) + 1 = 0. \quad (9) \end{aligned}$$

Now the equation of the true ellipse referred to the focus (*i. e.*, the position of the principal star) as origin is

$$\frac{(x+ae)^2}{a^2} + \frac{y^2}{b^2} - 1 = 0 \quad (10)$$

the x-axis coinciding with the major axis of the ellipse. Let us turn this axis back through the angle ω, to make it coincide with the line of nodes, by substituting for x and y, respectively, the values $x\cos\omega + y\sin\omega$, and $-x\sin\omega + y\cos\omega$, and equation (10) becomes

$$\frac{(x\cos\omega + y\sin\omega + ae)^2}{a^2} + \frac{(-x\sin\omega + y\cos\omega)^2}{b^2} - 1 = 0. \quad (11)$$

Equations (11) and (9) are necessarily identical since each represents the same ellipse referred to the same origin and the same axes; therefore the coefficients of the like powers of x and y must be proportional. Let f be the factor of proportionality. Then we shall have:

$$f\left(\frac{\cos^2\omega}{a^2} + \frac{\sin^2\omega}{b^2}\right) = A\cos^2\omega + B\sin^2\Omega + H\sin 2\Omega \quad (12)$$

$$f\left(\frac{\sin^2\omega}{a^2} + \frac{\cos^2\omega}{b^2}\right) = (A\sin^2\Omega + B\cos^2\Omega - H\sin 2\Omega)\cos^2 i \quad (13)$$

$$f\left(\frac{1}{a^2}-\frac{1}{b^2}\right)\sin 2\omega=(-A\sin 2\Omega+B\sin 2\Omega+2H\cos 2\Omega)\cos^2 i \quad (14)$$

$$f\frac{e\cos\omega}{a}=G\cos\Omega+F\sin\Omega \quad (15)$$

$$f\frac{e\sin\omega}{a}=(-G\sin\Omega+F\cos\Omega)\cos i \quad (16)$$

$$f(e^2-1)=+1. \quad (17)$$

From (17) we find

$$f=-\frac{1}{1-e^2}$$

and hence, introducing the semiparameter, $p=b^2/a$, the relation

$$f=-\frac{1}{1-e^2}=-\frac{a^2}{b^2}=-\frac{a}{p}, \text{ or, } a=\frac{p}{1-e^2}. \quad (17a)$$

We may now write (16) and (15) in the forms

$$\frac{e}{p}\sin\omega=-(F\cos\Omega-G\sin\Omega)\cos i$$
$$\frac{e}{p}\cos\omega=-(F\sin\Omega+G\cos\Omega). \quad (18)$$

Twice the product of equations (18) is

$$\frac{e^2}{p^2}\sin 2\omega=(F^2\sin 2\Omega-G^2\sin 2\Omega+2FG\cos 2\Omega)\cos i \quad (19)$$

and equation (14) may be written

$$\frac{e^2}{p^2}\sin 2\omega=(-A\sin 2\Omega+B\sin 2\Omega+2H\cos 2\Omega)\cos i; \quad (20)$$

hence

$$(F^2-G^2+A-B)\sin 2\Omega+2(FG-H)\cos 2\Omega=0. \quad (21)$$

Subtracting (13) from (12) and substituting for $\frac{1}{a^2}-\frac{1}{b^2}$ its equal $\frac{e^2}{p^2}$ we have

$$\frac{e^2}{p^2}\cos 2\omega=A\cos^2\Omega+B\sin^2\Omega+H\sin 2\Omega$$
$$-(A\sin^2\Omega+B\cos^2\Omega-H\sin 2\Omega)\cos^2 i. \quad (22)$$

The difference of the squares of the two equations (18) gives another value of $\frac{e^2}{p^2}\cos 2\,\omega$. Equating the two values and solving for $\cos^2 i$ we obtain

$$\cos^2 i = \frac{(F^2 - B)\sin^2\Omega + (G^2 - A)\cos^2\Omega + (FG - H)\sin 2\,\Omega}{(F^2 - B)\cos^2\Omega + (G^2 - A)\sin^2\Omega - (FG - H)\sin 2\,\Omega}. \qquad (23)$$

It is obvious from the forms of the numerator and denominator of this equation that if we put $\cos^2 i = \frac{N}{D}$ and therefore

$\tan^2 i = \frac{D-N}{N} = \frac{D+N}{N} - 2$ we shall have

$$\tan^2 i = \frac{F^2 + G^2 - (A+B)}{N} - 2. \qquad (24)$$

The first member of (13) may be written $\frac{e^2}{p^2}\sin^2\omega - \frac{1}{p^2}$ and the equation

$$\frac{e^2}{p^2}\sin^2\omega - \frac{1}{p^2} = (A\sin^2\Omega + B\cos^2\Omega - H\sin 2\,\Omega)\cos^2 i. \qquad (25)$$

Squaring (16) and substituting for $\frac{f^2}{a^2}$ its equal $\frac{1}{p^2}$ we find

$$\frac{e^2}{p^2}\sin^2\omega = (F^2\cos^2\Omega + G^2\sin^2\Omega - FG\sin 2\,\Omega)\cos^2 i; \qquad (26)$$

and from (25) and (26)

$$\frac{1}{p^2} = [(F^2 - B)\cos^2\Omega + (G^2 - A)\sin^2\Omega - (FG - H)\sin 2\,\Omega]\cos^2 i. \qquad (27)$$

Substituting the value of $\cos^2 i$ from (23) we have

$$\frac{1}{p^2} = (F^2 - B)\sin^2\Omega + (G^2 - A)\cos^2\Omega + (FG - H)\sin 2\,\Omega = N; \qquad (28)$$

therefore (24) may be written

$$\frac{2}{p^2} + \frac{\tan^2 i}{p^2} = F^2 + G^2 - (\mathrm{A} + \mathrm{B}). \qquad (29)$$

Writing for $\sin^2\Omega$ and $\cos^2\Omega$ in (28) the corresponding functions of 2Ω we find

$$\frac{2}{p^2} = F^2 + G^2 - (A + B) - (F^2 - B)\cos 2\Omega + (G^2 - A)\cos 2\Omega + 2(FG - H)\sin 2\Omega. \quad (30)$$

Therefore from (29) we have

$$\frac{\tan^2 i}{p^2} = (F^2 - G^2 + A - B)\cos 2\Omega - 2(FG - H)\sin 2\Omega. \quad (31)$$

Multiply (31) by $\sin 2\Omega$ and (21) by $\cos 2\Omega$ and subtract the latter result from the former. Then

$$\frac{\tan^2 i}{p^2}\sin 2\Omega = -2(FG - H). \quad (32)$$

Next multiply (31) by $\cos 2\Omega$ and (21) by $\sin 2\Omega$ and add the products. We have

$$\frac{\tan^2 i}{p^2}\cos 2\Omega = F^2 - G^2 + A - B. \quad (33)$$

Equations (33), (32), (29), (18), and (17a) define the geometric elements of the orbit in terms of the known constants derived from the measures with the micrometer.

To complete the solution analytically the period P and the time of periastron passage T will be found from the mean anomalies M computed from the observations by taking the ephemeris formulae on page 73 in the reverse order. Every M will give an equation of the form

$$M = \frac{360°}{P}(t - T), \text{ or, } M = \mu t + \epsilon, \text{ where } \epsilon = -\mu T$$

and the two unknowns P and T will be computed from all the equations by the method of least squares.

GLASENAPP'S MODIFICATION OF KOWALSKY'S METHOD

In theory, Kowalsky's method leaves nothing to be desired; given accurate measures it will lead to definitive results. But the measures of a double star, as we know, are affected by errors that are at present unavoidable, and, until means shall be devised to eliminate these more completely than we are now able to do, it will be more practical to adopt Glasenapp's suggestion and derive the five constants of the equation of the ellipse by his graphical method. Then we may apply Kowalsky's formulae, as before, to find the geometric elements of the orbit.

Glasenapp[3] assumes the apparent ellipse to have been drawn. Let its equation be, as before,

$$Ax^2 + 2Hxy + By^2 + 2Gx + 2Fy + 1 = 0. \quad (4)$$

Put $y = 0$; then the roots of the resulting equation

$$Ax^2 + 2Gx + 1 = 0$$

will be the abscissae of the points of the orbit on the x=axis. If we represent these roots by x_1 and x_2, we have, by the theory of equations,

$$A = \frac{1}{x_1 x_2}, \text{ and } G = -\frac{x_1 + x_2}{2x_1 x_2}. \quad (34)$$

Similarly, if we put $x = 0$, we have the equation

$$By^2 + 2Fy + 1 = 0,$$

whose roots will be the ordinates of the points of the orbit on the y=axis. From these we obtain

$$B = \frac{1}{y_1 y_2}, \text{ and } F = -\frac{y_1 + y_2}{2y_1 y_2}. \quad (35)$$

Therefore, direct measurement of the distances from the principal star to the intersections of the apparent ellipse with the axes of x and y, care being taken to regard the algebraic signs, will give the four constants A, B, G, and F. H, the remaining constant, is derived from

$$H = -\frac{Ax^2 + By^2 + 2Gx + 2Fy + 1}{2xy}. \quad (36)$$

Measure the coordinates of several points on the apparent ellipse, choosing such as will make the product xy as large as possible. Each set, substituted in (36) will give a value of H.

[3] *Monthly Notices Royal Astronomical Society* 49, 276, 1889.

The accordance of the separate values will depend upon the care with which the ellipse has been drawn, and the mean of all the values should be adopted.

Glasenapp's modifications practically convert Kowalsky's analytical method into a graphical one for the values of P and T, as well as the constants of the general equation which define the purely geometrical elements, may be determined by measures of the apparent ellipse. It is most convenient to make the measures for P and T with the aid of a planimeter as follows:

The position of the periastron point P is at that end of the diameter of the apparent ellipse drawn through the origin S which is nearest the origin, for this diameter is clearly the projection of the line of apsides of the true orbit. Having determined the constant of areal velocity (c) from the portion of the ellipse covered by the observations, we measure the areas of two sectors, PSt, and PSt', where t and t' represent observed positions on either side of P. Divide these areas by c and apply the quotients with the appropriate signs to the times corresponding to t and t'. The two resulting values of T, the *time* of periastron passage, should agree closely. More points than two may, of course, be used and the mean of all the values for T adopted. Similarly, the area of the entire ellipse divided by c gives the value of the revolution period. Since all the areas are simply relative it is not necessary to know the unit of area.

ZWIERS'S METHOD

Many methods have been published that enable the computer to derive the elements of the true orbit from graphical constructions. It is impossible to discuss them all in this chapter, and it is, fortunately, unnecessary. The *crux* of our problem is the construction of the apparent ellipse; when this has been accomplished, almost any of the methods which have been proposed will give satisfactory preliminary elements, provided the ellipse is a fairly open one. If it is very narrow and greatly elongated, none of the ordinary methods is entirely satisfactory. I have selected Zwiers's [4] method for presentation here because it is as simple as any and is one which I have found very convenient.

Zwiers assumes the apparent ellipse to have been drawn. Since it is the projection of the true orbit, the diameter which passes through the primary star's position S is the projection of the major axis of the true orbit and its conjugate is the projection of the minor axis. Further, if P is that extremity of the diameter through S which is nearest S it will be the projection of the point of periastron passage in the true orbit. Therefore, letting C represent the center of the ellipse, the ratio CS/CP will be the eccentricity, e, of that orbit, since ratios are not changed by projection.

Let $K = \frac{1}{\sqrt{1 - e^2}}$ be the ratio of the major to the minor axis in the true orbit; then, if all of the chords in this orbit parallel to the minor axis are increased in the ratio $K:1$ the ellipse will be transformed into Kepler's eccentric circle. Consequently, if in the *apparent* ellipse all ordinates parallel to the conjugate diameter, described above, are prolonged in the ratio $K:1$ we shall have another conic which may be called the *auxiliary ellipse.* It will evidently be the projection of the eccentric circle.

The major axis of the auxiliary ellipse will be a diameter of the eccentric circle and therefore equal to the major axis of the true orbit, and its position will define the line of nodes, since the nodal line must be parallel to the only diameter not shortened by projection. Designate the semi-major and semi-minor axes of the auxiliary ellipse by α and β respectively; then the ratio $\beta:\alpha$ is the cosine of the inclination of the orbit plane to the plane of projection. Again, the angle ω' between the major axis of the auxiliary ellipse and the diameter $PSCP'$ of the apparent orbit is the projection of the angle ω, the angle between node and periastron in the true orbit. Therefore

$$\tan \omega = \frac{\tan \omega'}{\cos i} = \frac{\alpha}{\beta} \tan \omega'. \quad (37)$$

[4] *A. N. 139*, p. 369, 1896. Professor H. N. Russell independently worked out a method based upon the same geometric concept. *A. J. 19*, p. 9, 1898.

Finally P and T are found by areal measures in the apparent ellipse in the manner already described.

The conjugate diameter required in Zwiers's construction may be drawn most easily by first drawing *any* chord of the ellipse parallel to $PSCP'$, the projected major axis. The diameter through the middle point of this chord is the conjugate required. If desired, advantage may also be taken of the fact that the conjugate diameter is parallel to the tangents to the ellipse at the points P and P'. The rectangular axes of the auxiliary ellipse may be found by trial or by the following construction:

Let

$$\frac{x^2}{(a')^2} + \frac{y^2}{(b')^2} = 1$$

be the equation of the apparent ellipse referred to its conjugate diameters. The equation of the auxiliary ellipse referred to the same axes will be

$$\frac{x^2}{(a')^2} + \frac{y^2}{(b')^2} = 1.$$

The axes are therefore also conjugate diameters of the auxiliary ellipse. At the extremity P of the diameter a' ($PSCP'$), erect two perpendiculars, PA and PB, to the tangent to the ellipse at this point and make each equal in length to Kb'. Through the extremities of the two perpendiculars and the center C of the apparent ellipse pass a circle. It will cut the tangent in two points, A' and B'. The lines $A'C$ and $B'C$ will give the directions of the two rectangular axes required, the major axis lying in the acute, the minor axis in the obtuse angle between the diameters α' and Kb'.

Instead of actually constructing the auxiliary ellipse it will generally be easier to derive the elements directly from measures of the apparent ellipse with the aid of simple formulae based upon the analytical solution of the construction. Thus:

Let e, a' and b' again represent respectively, the eccentricity, and the projected major and minor axes of the orbit, and let x_1 and x_2 be the position angles of a' and b'. To avoid ambiguity let x_1 be the position angle of the principal star as viewed from the center of the apparent ellipse and let x_2 be so taken that $(x_1 - x_2)$ is an acute angle. Also, compute as before, $K = \frac{1}{\sqrt{1 - e^2}}$ and $b'' = Kb'$. Then the relation between the rectangular axes 2α and 2β of the auxiliary ellipse and the conjugate diameters $2a'$ and $2b''$ are given by the equations

$$\alpha^2 + \beta^2 = a'^2 + b''^2$$
$$\alpha\beta = a' b'' \sin (x_1 - x_2) \quad (38)$$

the sine being considered positive.

The coordinates of any point on the auxiliary ellipse with respect to the axes 2α and 2β may be written in the form $\alpha \cos \phi'$, $\beta \sin \phi'$. Let $\alpha \cos (\omega)$, $\beta \sin (\omega)$ be the coordinates of the extremity of the a'-diameter; then we shall have

$$a'^2 = \alpha^2 \cos^2 (\omega) + \beta^2 \sin^2 (\omega) \quad (39)$$

and

$$\tan (\omega) = \pm \sqrt{\frac{\alpha^2 - a'^2}{a'^2 - \beta^2}} \quad (40)$$

in which the sign of $\tan (\omega)$ is to be the same as that of $(x_1 - x_2)$. But ω', the projection of ω is related to (ω) by the equation

$$\tan \omega' = \frac{\beta}{\alpha} \tan (\omega) \quad (41)$$

that is $(\omega) = \omega$ and $\Omega = (x_1 - \omega')$.

The angle ω obviously may have either of two values differing by 180°; that value is to be taken which will make Ω less than 180°.

Zwiers counts all angles in these formulae in the direction of increasing position angles.

The practical procedure may therefore be stated as follows: Construct the apparent ellipse and the diameter b' conjugate to a'; measure e, a', b', x_1 and x_2; compute $K = \frac{1}{\sqrt{1 - e^2}}$, $b'' = Kb'$, and find α and β from

$$(\alpha \pm \beta)^2 = a'^2 + b''^2 \pm 2a' b'' \sin (x_1 - x_2)$$

the sine being taken positive.

Then

$$a = \alpha$$

$$\cos i = \frac{\beta}{\alpha}$$

$$\tan \omega = \pm \sqrt{\frac{\alpha^2 - \alpha'^2}{\alpha'^2 - \beta^2}},$$

the sign of tan ω being taken the same as that of $(x_1 - x_2)$, and of the two values of ω that one which makes Ω less than 180°.

Next we have

$$\tan \omega' = \frac{\beta}{\alpha} \tan \omega, \ \Omega = (\chi' - \omega'),$$

and finally deduce the values of P and T from area measurement, as in the Glasenapp-Kowalsky method.

THE ORBIT OF A 88

The binary system, A 88 (R. A. 18h. 33m. 9s.; Decl. −3° 17′; magnitudes, 6.9, 7.1), which was discovered with the thirty-six-inch telescope in 1900, will be used to illustrate the orbit methods which have just been described. All of the observations of this pair have been made by the writer, and from them it was seen that the period of revolution must be approximately twelve years, for in 1912 the companion was again

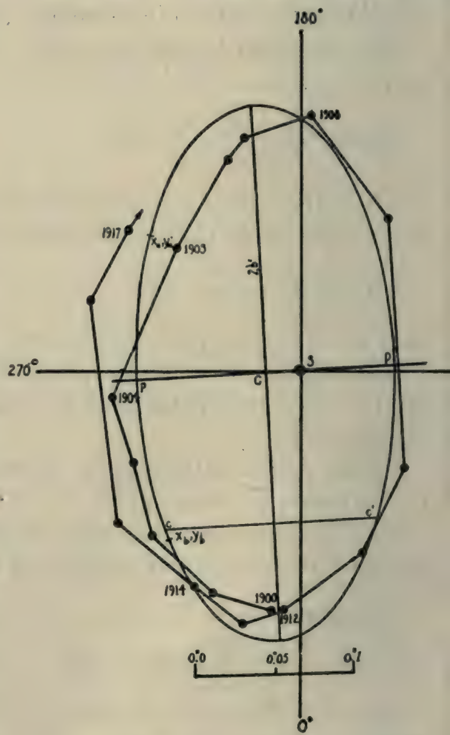

FIGURE 3. The Orbit of A 88

nearly in the position it occupied at the time of discovery. The following orbit was computed at that time by the Glasenapp-Kowalsky method and is purposely not revised, although the observed angular motion since then amounts to 128°, for it will be of interest to see how closely a preliminary orbit of a pair so difficult to measure may be expected to represent later observations. The maximum separation of the two components is only 0.17″.

The observations to date are given in the first three columns of the following table. The fourth column shows the number of measures (on different nights) on which each position rests, and the last two columns give the residuals, observed minus computed position angles and distances. One or two of the angle residuals are too large to be at all satisfactory; but, in estimating them, the extreme closeness of the pair must be kept in mind; a residual of 24° in angle corresponds to a displacement of the companion of but half the thickness of the micrometer thread.

Measures and Residuals for A 88

Date	*Angle* °	*Dist.*″	*n*	*O–C*	
				$\Delta\theta$°	$\Delta\rho$″
1900.46	353.2	0.14	3	− 1.8	−0.01
1901.56	338.3	0.14	3	− 0.9	−0.02
1902.66	318.1	0.12	3	+ 0.6	−0.01
1903.46	293.6	0.11	4	− 0.8	±0.00
1904.53	278.4	0.14	4	+24.1	+0.03
1905.53	224.8	0.12	4	+ 0.1	−0.01
1906.48	199.1	0.13	4	− 7.1	−0.03
1907.30	193.5	0.14	1	− 0.5	−0.03
1908.39	178.1	0.15	3	+ 3.3	±0.00
1909.67	150.4	0.10	2	+14.7	+0.03
1910.56	47.0	0.11	2	− 0.7	+0.03
1911.55	18.7	0.15	1	+ 6.9	+0.01
1912.57	356.1	0.15	3	+ 0.8	±0.00
1914.54	333.9	0.15	5	+11.2	+0.01
1915.52	306.4	0.15	3	+11.0	+0.04
1916.76	248.4	0.14	2	− 1.9	+0.02
1917.65	228.1	0.14	2	+ 3.2	+0.02

All of the measures to 1912 inclusive were plotted, using a scale of three inches to 0.1″, and, after repeated trials, the ellipse shown in the diagram was drawn. It represents the observation points fairly and satisfies the law of areas closely. Applying the Glasenapp-Kowalsky method, we first measure the intercepts of the ellipse with the axes of coordinates, and the coordinates of two selected points for the value of H, counting the end of the x-axis at 0°, and of the y-axis at 90°, positive. The measures are (in inches on the original drawing):

$$x_1 = +4.98,\ y_1 = +1.77,\ x_a = -2.55,\ y_a = -2.86$$
$$x_2 = -4.73,\ y_2 = -3.12,\ x_b = +3.17,\ y_b = -2.49.$$

Therefore we have

$$x_1 x_2 = -23.5554,\quad y_1 y_2 = -5.5224,\quad x_a y_a = +7.2930,$$
$$x_1 + x_2 = +0.25,\quad y_1 + y_2 = -1.35,\quad x_b y_b = -7.8933,$$
$$x_a^2 = 6.5025,\quad y_a^2 = 8.1796$$
$$x_b^2 = 10.0489,\quad y_b^2 = 6.2001$$

from which to compute the five constants of the equation of the ellipse. We find

$$A = \frac{1}{x_1 x_2} = -0.04245$$

$$B = \frac{1}{y_1 y_2} = -0.18108$$

$$F = -\frac{y_1 + y_2}{2\, y_1 y_2} = -0.12223$$

$$G = -\frac{x_1 + x_2}{2\, x_1 x_2} = +0.00531.$$

From these values and the coordinates x_a, y_a, we obtain

$$H = -\frac{Ax^2 + By^2 + 2Gx + 2Fy + 1}{2xy} = +0.00584,$$

and, similarly, from the coordinates x_b, y_b,

$$H = +0.00590,$$

and adopt the mean, $+0.00587$.

Combining these constants, we have,

$FG = -0.00065;\quad F^2 = +0.01494;\quad G^2 = +0.00003;$
$-2(FG - H) = +0.013;\quad F^2 - G^2 + A - B = +0.15354;$
$F^2 + G^2 - (A + B) = +0.23850.$

The solution of equations (32), (33), (29), (18), and (17a) then proceeds as follows:

l. $\frac{\tan^2 i}{p^2} \sin 2\Omega$	8.11528
l. $\frac{\tan^2 i}{p^2} \cos 2\Omega$	9.18622
l. $\tan 2\Omega$	8.92906
2Ω	4.85°
Ω	2.4°
l. $\cos 2\Omega$	9.99844
l. $\frac{\tan^2 i}{p^2}$	9.18778
$\frac{\tan^2 i}{p^2}$	+ 0.15409
From (29) $\frac{2}{p^2} + \frac{\tan^2 i}{p^2}$	+ 0.23850
$\frac{2}{p^2}$	+ 0.08441
$\frac{1}{p^2}$	+ 0.04220
l. $\frac{1}{p^2}$	8.62536
l. p^2	1.37464
l. p	0.68732
l. $\tan^2 i$	0.56242
l. $\tan i$	0.28121
i	62.4°
$\cos i$	9.66586

log F	9.08717_n	log G	7.72482
sin Ω	8.62557	sin Ω	8.62557
cos Ω	9.99961	cos Ω	9.99961
(1) l. $F\cos\Omega$	9.08678_n	(3) l. $G\sin\Omega$	6.35039
(2) l. $F\sin\Omega$	7.71264_n	(4) l. $G\cos\Omega$	7.72443
(1)	−0.12212	(3)	+0.00022
(2)	−0.005160	(4)	+0.005302
(1) − (3)	−0.12234	(2) + (4)	+0.000142
l.−[(1)−(3)]	9.08757	l.−[(2)+(4)]	6.15229_n
cos i	9.66586	l. p	0.68732
l. p	0.68732		
log $e\sin\omega$	9.44075	log $e\cos\omega$	6.83961
l. tan ω	2.60114_n		
ω	90.1°		
sin ω	0.00000		
log e	9.44075		
e	0.276		
e^2	0.07618		
$1 - e^2$	0.92382		
l.$(1 - e^2)$	9.96559		
l. p	0.68732		

$$\text{l.}\ \frac{p}{(1-e^2)} = \log a\ 0.72173$$

a 5.269 inches
= 0.176″

From the diagram it is obvious that the companion passed its periastron point between the dates of observation 1909.67 and 1910.56; but the measures made in 1908 and 1912 were regarded as more reliable than these and were accordingly used to determine the time of periastron passage. The constant of areal velocity (in units of the planimeter scale) had been found to be 0.205. Drawing radii to the points P and 1908.39 and 1912.57, the areas of the two resulting sectors were, in terms of the same unit, respectively, 0.34440 and 0.50225. Hence the time intervals between these two dates and the date of periastron passage were, respectively, +1.68 years, and −2.45

years, giving for T, the two values, 1910.07 and 1910.12. The mean, 1910.1, was adopted. The planimeter measures gave as the area of the entire ellipse, 2.4848, and the period, 12.12 years.

To solve the orbit by Zwiers's method, we begin by drawing the axis b' conjugate to a' ($PSCP'$). Draw the chord cc, parallel to $P'CSP$ and then draw the diameter through its middle point. This will be the required conjugate.

We now measure $CS=0.67$, $CP=2.45$, $a'=2.445$, $b'=5.050$; and the angles $x_1=92.6°$, and $x_2=3.6°$.

The ratio $CS{:}CP$ gives at once the value of the eccentricity, $e=0.273$, and from this we compute the value of $K=\dfrac{1}{\sqrt{1-e^2}}$ (in logarithms) 0.01682. Thence we find $b''=Kb'=5.2494$.

The computation then proceeds as follows:

$\log a'$	0.38828	$(a')^2$	5.9780
$\log b'$	0.72011	$(b'')^2$	27.5562
$\log 2$	0.30103		
$\text{l.} \sin(x_1 - x_2)$	9.99993	$[(a')^2+(b'')^2$	33.5342
$\text{l.}\ 2a'b''\sin(x_1-x_2)$	1.40935	α^2	27.5600
$2a'b''\sin(x_1-x_2)$	25.6653	β^2	5.9756
$(a')^2+(b'')^2$	33.5342	$\alpha^2-(a')^2$	21.5820
$(\alpha+\beta)^2$	59.1995	$(a')^2-\beta^2$	0.0024
$(\alpha-\beta)^2$	7.8689	$\log[\alpha^2-(a')^2]$	1.33409
$(\alpha+\beta)$	7.6942	$\log[(a')^2-\beta^2]$	7.38021
$(\alpha-\beta)$	2.8052	$\log\tan^2\omega$	3.93388
2α	10.4994	$\log\tan\omega$	1.96694
2β	4.8890	$\therefore \omega = 89.4°$	
α	5.2497	$\log\cos i$	9.66805
β	2.4445	$\log\tan\omega'$	1.63499
$\log\beta$	0.38819	ω'	88.7°
$\log\alpha$	0.72014	$\therefore \Omega = (x_1-\omega') = 3.9°$	
$\log\cos i$	9.66805		
$\therefore i = 62.25°$			
$a=\alpha=$	5.25 inches		
$=$	0.175″		

Assembling the elements we have the following:

	Glasenapp's Method	*Zwiers's Method*
$P =$	12.12 years	12.12 years
$T =$	1910.10	1910.10
$e =$	0.276	0.273
$a =$	0.176″	0.175″
$\omega =$	269.9°	270.6°
$i =$	± 62.4	± 62.25
$\Omega =$	2.4	3.9

Angles decreasing with the time.

In the formulae, all angles are counted in the direction of increasing position angles, whereas in the notation given on page 72 ω is counted from node to periastron *in the direction of motion of the companion*. Therefore, when as in this system the observed position angles decrease with advancing time, the value for ω derived from the formulae must be subtracted from 360°. In applying the formulae for computing the ephemeris of such a system, the anomalies are counted positive after periastron passage and negative before, just as in the case of direct motion (angles increasing with the time); $\cos i$ is counted as positive, and the angles $(\theta-\Omega)$ are taken in the quadrant $360° - (v+\omega)$. I have found this to be the simplest and most satisfactory method of procedure in every case where the angles decrease with the time. In orbits with direct motion the value of ω is used as given directly by the formulae and the angles $(\theta-\Omega)$ are taken in the same quadrant as the angles $(v+\omega)$.

CONSTRUCTION OF THE APPARENT ELLIPSE FROM THE ELEMENTS

It is sometimes desirable to be able to construct the apparent ellipse from the elements of the true orbit. This construction is easily and quickly effected in the following manner:

Take the point O, at the intersection of two rectangular axes, OX and OY, as the common center of the true and projected orbits. Draw the line $O\Omega$ making an angle equal to Ω with the line OX, counting from 0°. Lay off the angle ω from the line $O\Omega$, starting from the extremity Ω between 0° and 180° and *proceeding in the direction of the companion's motion* (clockwise, that is, if the position angles decrease with the time, counter-clockwise, if they increase with the time). This will give the direction of the line of apsides, AOP, in the true orbit.

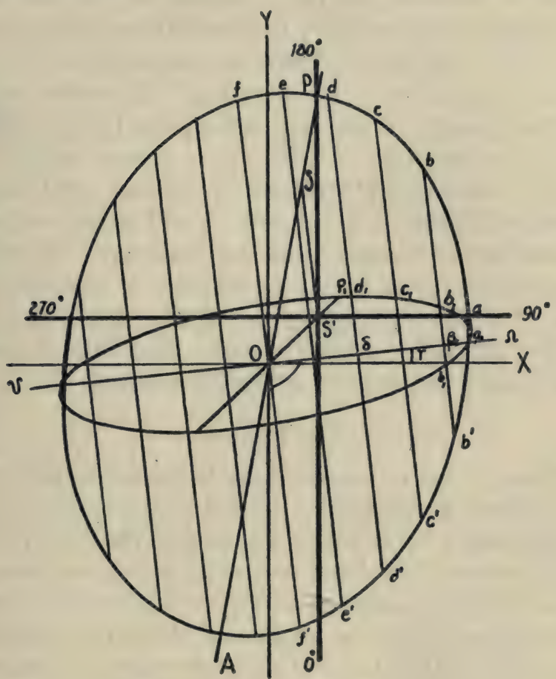

FIGURE 4. The True and Apparent Orbits of a Double Star (after See)

Upon this line lay off OS, equal to ae, the product of the eccentricity and the semi-axis of the orbit, using any convenient scale, and OP and OA, each equal to a. The point S lies between O and P, and P is to be taken in the quadrant given by applying ω to Ω as described above. Having thus the major axis and the eccentricity, the true ellipse is constructed in the usual manner.

Now divide the diameter ℧$O\Omega$ of this ellipse into any convenient number of parts, *making the points of division symmetrical with respect to O*, and draw chords $b\beta b'$, etc. perpendicular to the line of nodes. Measure the segments $b\beta$, $\beta b'$, etc. and multiply the results by $\cos i$. The products will evidently be the lengths of the corresponding segments βb_1, βb_2, etc., in the projected ellipse, and the curve drawn through the points b_1, c_1, d_1, . . . will be the desired apparent orbit.

To find the position of the principal star in the apparent ellipse draw through S a line perpendicular to the line of nodes, and find its intersection S' with an arc drawn with O as a center and a radius equal to $OS \cos i$. This is the point required. Lines through S' parallel to OX and OY will be the rectangular axes to which position angles in the apparent orbit are referred, and the position angle of the companion at any particular epoch may be obtained by laying off the observed position angle. The line OS' extended to meet the ellipse defines P', the projection of the point of periastron passage.

DIFFERENTIAL CORRECTIONS

If sufficient care is exercised in the construction of the apparent ellipse, methods like those described will, as a rule, give a preliminary orbit which will satisfy the observed positions within reasonable limits and which will approximate the real orbit closely enough to serve as the basis for a least squares solution. It may be remarked that a satisfactory representation of the observed positions does not necessarily imply a correct orbit when the arc covered by the observations is comparatively small. The percentage of error inherent in double star measures in so great that, if the observed arc is less than 180°, it will generally be possible to draw several very different ellipses each of which will satisfy the data of observation about equally well. *In general, it is not worth while to compute the orbit of a double star until the observed arc not only exceeds 180°, but also defines both ends of the apparent ellipse.*

Many computers are content with a preliminary orbit; but it is advisable to correct these elements by the method of least squares whenever the data are sufficient for an investigation of the systematic errors of observation.

The position angle is a function of the six elements

$$\Omega,\ i,\ \omega,\ e\ (= \sin\phi),\ T \text{ and } \mu = \frac{360^\circ}{P}$$

and the required differential coefficients for the equations of condition can be computed with all necessary accuracy from the approximate formula

$$a\Delta\Omega + b\Delta i + c\Delta\omega + d\Delta\phi + \epsilon\Delta M_\circ + f\Delta\mu + (C_\circ - O_\circ) = 0 \quad (43)$$

where $\Delta\Omega$, etc., are the desired corrections to the elements, $M_\circ = \mu(t-T)$, $(C_\circ - O_\circ)$ is the residual, computed minus observed position angle, and a, b, c, d, ϵ, f, are the partial differential coefficients.

These are derived from the equations

$$M = \mu\,(t - T) = E - e\sin E$$

$$\tan \tfrac{1}{2} v = \sqrt{\frac{1+e}{1-e}}\tan \tfrac{1}{2} E$$

$$\tan\,(\theta - \Omega) = \cos i \tan\,(v + \omega)$$

and their values are

$$a = 1$$

$$b = -\sin i \tan\,(v + \omega)\cos^2(\theta - \Omega)$$

$$c = \cos^2(\theta - \Omega)\sec^2(v + \omega)\cos i$$

$$d = \frac{2 - e\cos E - e^2}{(1 - e\cos E)^2}\sin E \cdot C$$

$$\epsilon = \frac{\cos\phi}{(1 - e\cos E)^2} \cdot C$$

$$f = (t - T)\cdot\epsilon.$$

To facilitate the solution, the coefficients in the equations of condition should be reduced to the same numerical order by the introduction of suitable multipliers.

The corresponding differential equation for the distance correction may be derived by differentiating the formula

$$\rho = a\,(1 - e\cos E)\cos\,(v + \omega)\sec\,(\theta - \Omega),$$

but it is customary to compute the correction for a, the semi-axis major, directly from the residuals in distance after the remaining elements have been corrected by the aid of equation (43).[5]

Equation (43) is strictly applicable only when the residuals in angle are independent of the angular distance between the companion and primary star. When the eccentricity and the inclination of the orbit are both small this condition is approximately realized, but when either of these elements is large, it is clear that the space displacement produced by a given error in angle will vary greatly in different parts of the orbit, and the equation must be modified so that the solution will make the sum of the squares of the space displacements a minimum rather than that of the angle residuals. This can, in general, be effected with sufficient accuracy by multiplying the values

[5] Comstock has just published (The Orbit of Σ 2026. By George C. Comstock. *Astronomical Journal*, vol. *31*, p. 33, 1918) expressions which are more convenient in numerical application than those given in the text and which have the additional advantage of permitting the equations derived for $\Delta\rho$ to be combined into a single solution with those for $\Delta\theta$ after Δa has been eliminated from the one group and $\Delta\Omega$ from the other.

Write the two groups in the forms:

(for θ) $A\,\Delta\Omega + B\Delta\omega + C\Delta i + D\Delta\phi + F\mu\Delta T + G\Delta\mu + (C - O) = 0.$

(for ρ) $h\Delta a + b\Delta\omega + c\Delta i + d\Delta\phi + f\mu\Delta T + g\Delta\mu + (C - O) = 0.$

As in equation (43) the eccentric angle ϕ, defined by $\sin\phi = e$, is introduced as an element instead of the eccentricity e. To make the two groups of equations homogeneous, $(C - O)$ in the equations for θ must be expressed in circular measure, that is, instead of $\Delta\theta$ we must write $\rho\Delta\theta \div 57.^\circ 3$. The corresponding corrections are taken into account in the differential coefficients which follow. For convenience form the auxiliary quantities

$$m\ (= \rho/57.^\circ 3) = [8.2419]\,\rho,\quad \kappa = (2 + \sin\phi\cos v)\sin E$$
$$\sigma = -m\tan i\sin(\theta - \Omega)\cos(\theta - \Omega).$$

Then we have

$$A = +m \qquad h = +\frac{\rho}{a}$$
$$B = +m\left(\frac{r}{\rho}\right)^2\cos i \qquad b = +\sigma\sin i$$
$$C = +\sigma \qquad c = +\sigma\tan(\theta - \Omega)$$
$$D = +B\frac{a}{r}\kappa \qquad d = +b\frac{a}{r}\kappa - m\frac{a}{r}\cos\phi\cos v$$
$$F = -B\left(\frac{a}{r}\right)^2\cos\phi \qquad f = -b\left(\frac{a}{r}\right)^2\cos\phi - m\left(\frac{a}{r}\right)^2\sin\phi\sin E$$
$$G = -F(t - T) \qquad g = -f(t - T)$$

The solution of these equations gives Δa in seconds of arc and the other unknowns in degrees. If θ decreases with the time, count the anomalies positive before periastron, negative after periastron passage.

$(C_0 - O_0)$ by factors proportional to the corresponding observed distances.

SPECIAL CASES

The methods of Kowalsky and of Zwiers and all other methods based upon the construction of the apparent ellipse fail when the inclination of the orbit plane is 90°; for then the apparent ellipse is reduced to a straight line and the observed motion is entirely in distance, the position angle remaining constant except for the change of 180° after apparent occultation. Such a limiting case is actually presented by the system 42 *Comae Berenices*, and many other systems are known in which the inclination is so high that the apparent orbit is an extremely narrow ellipse, differing but little from the straight line limit. Special methods, based chiefly upon the curve of the observed distances, must be devised in such cases. Advantage may also be taken of peculiarities in the apparent motion in some systems to obtain approximate values of one or more of the orbit elements.

In addition to the inclination (=90°), the element Ω is known from the conditions in the case of a system like 42 *Comae*, for this must be the mean of the observed position angles. The remaining elements must be determined from the curve representing the observed distances. Let us assume that the observed distances have been plotted against the times and that the most probable curve has been drawn through the plotted points. The revolution period may then be read directly from the curve, the accuracy of the determination depending upon the number of whole revolutions included in the observations as well as upon the precision of the measures. In general, the elements e, the eccentricity, and T the time of periastron passage, are as easily determined.

Let PK_2P_1, in Fig. 5, represent the true orbit, and $BS'C'B'$, its projection. The point C' is known for BC' must equal half the amplitude of the curve of distances. The point S' is known, since it is the origin from which the distances are measured, *i. e.*, the position of the primary star. It is also evident that the points on the curve of distances which correspond to the points P and P_1 in the true orbit must be separated by precisely half of the revolution period and that their ordinates, measured from the line CEE', must be equal in length and of opposite sign. The point corresponding to periastron must lie on the same side of this line as S', and on the steeper branch of the curve. In practice these two points are most readily found by cutting a rectangular slip of paper to a width equal to half that

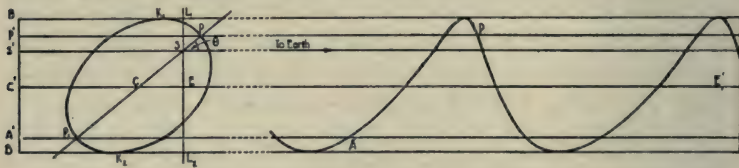

FIGURE 5. Apparent and True Orbits, and Interpolating Curve of Observed Distances for a Binary System in which the Inclination is 90°

of the period on the adopted scale and sliding it along the curve until the edges, kept perpendicular to the line CEE', cut equal ordinates on the curve.

When P has been found on the curve, draw the line PP' parallel to CEE'. The value of e follows at once from the ratio $C'S' : C'P'$.

There remain the two elements a and ω; and these cannot be derived quite so simply. The following process for their determination is due to Professor Moulton.[6]

Let θ be the angle between the line of apsides and the line to the Earth. It is equal to $90° - \omega$. Then in the figure, we have

$$S'C' = SE = ae \sin \theta. \quad (44)$$

This gives one relation between the three elements a, e and θ, for the length SE is known.

To find another let us take the equation of the ellipse with the origin at its center, assuming the X-axis to be, as usual, coincident with the major axis. We have

$$\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = 1.$$

[6] Kindly sent to me by letter.

Remove the origin to the focus, S, and the equation becomes

$$\frac{(x_1 - ae)^2}{a^2} + \frac{y_1^2}{a^2(1 - e^2)} = 1$$

which we shall write in the form

$$(x_1 - ae)^2(1 - e^2) + y_1^2 - a^2(1 - e^2) = 0. \qquad (45)$$

Now rotate the axes backward through the angle θ, thus making the X-axis point toward the Earth, the transformation equations being

$$x_1 = x \cos \theta + y \sin \theta$$
$$y_1 = x \sin \theta + y \cos \theta.$$

We obtain

$$f(x_1 y) = (1 - e^2 \cos^2 \theta)\, x^2 + (1 - e^2 \sin^2 \theta) y^2 - 2e^2 \sin \theta \cos \theta\, xy$$
$$- 2ae(1 - e^2) \cos \theta\, x - 2ae(1 - e^2) \sin \theta\, y - a^2(1 - e^2)^2 = 0. \qquad (46)$$

Let $y = c$, be the equation of a line parallel to the X-axis and cutting the ellipse in two points. The x-coordinates of the points of intersection are given by

$$f(x_1 c) = (1 - e^2 \cos^2 \theta)\, x^2 - 2e \cos \theta\, [ec \sin \theta + a(1 - e^2)]\, x$$
$$+ (1 - e^2 \sin^2 \theta)\, c^2 - 2\,aec(1 - e^2) \sin \theta - a^2(1 - e^2)^2 = 0. \qquad (47)$$

To obtain the tangent K_1, L_1, we must impose the condition that the two roots of equation (47) are equal; that is, that

$$\frac{df}{dx} = 2(1 - e^2 \cos^2 \theta)\, x - 2e \cos \theta\, [ec \sin \theta + a(1 - e^2)] = 0. \qquad (48)$$

Substitute the value of x from (48) in (47), and solve for c which, by the conditions, is equal to EL ($= C'B'$) and is therefore a known quantity. After simplification we obtain

$$c^2 - 2\,aec \sin \theta - a^2(1 - e^2) = 0 \qquad (49)$$

which is the desired second relation between the three elements.

Combining (44) with (49), we have

$$a = \sqrt{\frac{c^2 - 2\,aec \sin \theta}{1 - e^2}} = \sqrt{\frac{(EL_1)^2 - 2SE \cdot EL_1}{1 - e^2}} = \sqrt{\frac{(C'B')^2 - 2S'C' \cdot C'B'}{1 - e^2}} \qquad (50)$$

and the value of ω ($= 90° - \theta$) follows at once from (44).

If we please, we may write a third relation, independent of the other two since it is dynamical and rests upon the law of areas, in the form,

$$S'P' = a(1 - e) \sin \theta. \qquad (50a)$$

Now let us write (44), (49), and (50a) in the forms

$$\begin{aligned} ae \sin \theta &= A \\ a\,(1 - e) \sin \theta &= B \qquad (50b) \\ a^2 (1 - e^2) + 2ae\,C \sin \theta &= C^2 \end{aligned}$$

and we find at once

$$\begin{aligned} e &= \frac{A}{A + B} \\ a &= \frac{\sqrt{C^2 - 2AC}}{\sqrt{1 - e^2}} \qquad (50c) \\ \sin \theta &= \frac{\sqrt{B^2 + 2AB}}{\sqrt{C^2 - 2AC}} \end{aligned}$$

where $A = SE = S'C'$, $B = S'P'$ and $C = EL_1 = C'B'$.

This solution fails only when the points C' and S' are coincident, that is, when the two elongation distances are equal, and this will only occur (1) when the true orbit is circular, or (2), when the major axis of the ellipse lies in the line of sight. The two cases may be distinguished by the fact that in the former the *time intervals* from apparent coincidence of the two stars to the elongations at either side will be equal, in the latter, unequal. When they are equal, the elongation distance is the radius, or semi-axis, a, of the true orbit, and any convenient epoch, for example that corresponding to apparent coincidence, may be adopted as origin in reckoning the times. The elements T and ω have, of course, no significance in this case. When the elongation times are unequal, the elongation distance gives the semi-axis minor, b, of the true orbit, the epoch of coincidence which falls in the shorter interval between successive elongations is the epoch of periastron, and ω equals $\Omega \pm 90°$. The element e cannot in this case be found by the direct method given above. Probably it may be derived from the dynamical relation $T_1/T_2 = f(a_1, e_1, \theta)$, where T_1 and T_2 are the epochs of the two elongations. The case will be a very rare one, and I have not attempted its solution.

When a preliminary set of elements has been derived by the methods described, improved values may be computed by the method of least squares, the equations of condition being obtained by differentiating the formula for the apparent distance, which in orbits of this character takes the form

$$\rho = r\cos(v+\omega) = a(1 - e\cos E)(\cos v + \omega).$$

We may write the equations of condition in the form

$$\Delta\rho = A\Delta a + B\Delta\omega + C\Delta\phi + D\Delta M_0 + D(t-T)\Delta\mu \quad (50d)$$

where

$$A = \frac{\rho}{a}$$

$$B = -r\sin(v+\omega)$$

$$C = \left(\frac{a}{r}\right)^2\left(\frac{\rho}{a}\right)(\sin\phi - \cos E) - \sin v\sin(v+\omega)$$

$$\left(1 + \frac{r}{a\cos^2\phi}\right)a\cos\phi$$

$$D = a\left[\left(\frac{a}{r}\right)^2\left(\frac{\rho}{a}\right)\sin\phi\sin E - \frac{\sin v\sin(v+\omega)}{\sin E}\right].$$

THE ORBIT OF ϵ EQUULEI

An excellent example of an orbit whose computation was made possible by taking advantage of the special features of the observed motion is that of ϵ *Equulei*, recently published by Russell.[7] The apparent orbit of this system is an extremely narrow and elongated ellipse. Fortunately the double star was discovered by Struve, in 1835, when the angular separation was only 0.35″. In later years the companion moved out to a maximum elongation of 1.05″ and then in again until now (1917) the pair cannot be resolved by any existing telescope. Plotting the distances (using mean places) against the times, Russell noted that the curve was practically symmetrical with

[7] *Astronomical Journal* 30, 123, 1917.

respect to the maximum separation point. It follows that the line of apsides in the true orbit must be approximately coincident with the line of nodes, or in other words that $\omega = 0$. Further, the mean of the position angles for a few years on either side of the time of elongation gives a preliminary value for the angle Ω, and the elongation time itself is the epoch of apastron passage, which may be taken in place of the epoch of periastron as one of the orbit elements. It is also apparent that the inclination of the orbit is very high and a preliminary value for this element may be assumed. This leaves the three elements, a, e and P, which Russell finds as follows:

Let y_1 = the maximum elongation distance
T' = the corresponding epoch (*i. e.*, apastron)
$y_{\prime\prime}$ = the distance at any other time $t_{\prime\prime}$
E = the corresponding eccentric anomaly in the true orbit.

Then we have

$$a(1 + e) = y_1,\ a\,(\cos E - e) = -y_{\prime\prime}$$
$$M = E - e\cos E, \text{ and } (t_{\prime\prime} - T) = 180^\circ - M,$$

which determine a and P in terms of e.

Assume values of e and compute a and P, repeating the process until those values result which represent the curve of the observed distances.

The preliminary elements obtained by these processes Russell corrects differentially, a, e, T and μ from the observed distances, i and Ω from the observed angles, ω $(=0)$ being assumed as definitely known.

SYSTEMS IN WHICH ONE COMPONENT IS INVISIBLE

Luminosity, as Bessel said long ago, is not a necessary attribute of stellar mass, and it may happen that one component of a double star system is either entirely dark or so feebly luminous as not to be visible in existing telescopes. If the orbit is one of short period and the inclination of its plane sufficiently high, the system may be detected by the spectroscope, by the methods to be discussed in the following chapter. In other instances the companion's presence may be revealed by a periodic variation in the bright star's proper motion, the path described by it upon the celestial sphere becoming a cycloid instead of the arc of a great circle. A system of the latter type is most readily detected when the proper motion is large, and it is, of course essential that the motion be accurately determined.

Variable proper motion was actually recognized in the stars *Sirius* and *Procyon*, about three-quarters of a century ago, and was explained by Bessel as the effect of the attraction of such invisible companions. Orbits, referring the motion of the bright star to the center of gravity of a binary system, were thereupon computed for these stars by C. A. F. Peters and A. Auwers. Bessel's hypothesis was proven to be correct by the subsequent discovery of a faint companion to *Sirius* by Alvan G. Clark (in 1861), and of a still fainter companion to *Procyon* by Schaeberle (in 1896). The relative orbit of the companion to *Sirius* has been computed from the micrometer measures, and the elements are consistent with those determined from the proper motion of the bright star. There is no question but that this will also prove to be the case in the system of *Procyon* when the micrometer measures permit an independent determination of its orbit.

Dark companions to β *Orionis*, α *Hydrae*, and α *Virginis* have also been suspected from supposed irregularities in the proper motions, but closer examination of the data has not verified the suspicion. Since cases of this kind will probably always be very exceptional, the formulae for their investigation will not be considered here. Those who are interested in their development are referred to the original memoirs.[8]

The presence of invisible companions in several well-known double star systems has also been suspected on account of observed periodic variations in the motion of one of the visible components. In one of these, ϵ *Hydrae*, the primary star was later found to be a very close pair whose components complete

[8] Bessel, *A. N. 22*, 145, 169, 185, 1845.

Peters, *A. N. 32*, 1, 17, 33, 49, 1851.

Auwers, *A. N. 63*, 273, 1865 and *Untersuchungen über veränderliche Eigenbewegung*, 1 Theil, Köningsberg, 1862; 2 Theil, Leipzig, 1868. See also *A. N. 129*, 185, 1892.

a revolution in about fifteen years, and Seeliger[9] has shown that the orbital motion in this close pair fully accounts for the irregularities observed in the motion of the more distant companion. Another of these systems, ζ *Cancri*, consists of three bright stars, two of which revolve about a common center in a period of approximately sixty years, while the third star revolves with this binary system in a much larger orbit. Seeliger has shown that the irregularities observed in the apparent motion of this third star may be explained on the hypothesis that it is accompanied by an invisible star, the two revolving about a common center in circular orbits with a period of eighteen years. The system would, then, be a quadruple one. There are irregularities in the observed motion of 70 *Ophiuchi* which are almost certainly due to the perturbations produced by a third body, but a really satisfactory solution of the orbit has not yet been published. Finally, Comstock[10] has just published a model investigation of the orbital motion in the system ζ *Herculis* from which he concludes that small irregularities in the areal velocity of the bright pair may be represented as the effect of an invisible companion to one component, having a periodic time of 18 years and an amplitude less than 0.1″. Comstock, however, points out that when the systematic errors of the observers are determined and allowed for, the orbit, without the assumption of a third body, "satisfies the observations within the limits of error commonly deemed satisfactory." The paper is an excellent example of the method in which systematic errors should be investigated in the computation of a definitive double star orbit.

It is probable that the invisible companion in such a system as that of ζ *Herculis* revolves, like the bright components, in an elliptic, rather than a circular orbit; and it is not at all improbable that the plane of this orbit is inclined at a greater or less angle to the plane of the orbit of the visible system. To determine the eccentricity and the inclination, however, would greatly complicate the problem and the precision of the observational data is not sufficient to warrant such refinements.

[9] *Astronomische Nachrichten* 173, 325, 1906.
[10] *Astronomical Journal* 30, 139, 1917.

In practice, it has been found satisfactory to assume that the invisible body moves in a circle in the plane of the orbit of the visible stars of the system. This assumption leaves but two elements to be determined, the period and the radius or semi-amplitude, and the formulae for these are quite simple. Comstock's formulae for the companion in the system of ζ *Herculis*, for example, are as follows:

Let θ, ρ, represent the polar coordinates of the visible companion referred to the primary star; ψ, r the corresponding coordinates of the center of gravity of the assumed system (*i. e.* the system comprised of the secondary bright star and its dark companion) referred to the same origin; and v, a, the coordinates of the visible companion referred to the center of mass of itself and its dark companion. Then we shall have from the geometrical relations involved,

$$\rho^2 = r^2 + a^2 + 2ar\cos(v - \psi)$$

$$\theta = \psi + \frac{a}{\rho}\sin(v - \psi). \quad (51)$$

If we assume that $\frac{a}{r}$ and $\frac{r}{\rho}$ are quantities whose squares are negligibly small, we have by differentiation

$$\rho^2\frac{d\theta}{dt} = r^2\frac{d\psi}{dt} + ar\cos(v-\psi)\left[\frac{dv}{dt} + \frac{d\psi}{dt}\right] - a\sin(v - \psi)\frac{dr}{dt}. \quad (52)$$

Since the assumed system is circular, a and $\frac{dv}{dt}$ are constant quantities. $r^2\frac{d\psi}{dt}$ is also a constant, and a is so small that, in the second member of the equation, we may write θ for ψ and ρ in place of r without sensible error. If, further, for brevity, we put $\frac{d\psi}{dt} = \frac{K}{\rho^2}$ and $k = \frac{dv}{dt}$, the equation takes the form

$$\rho^2\frac{d\theta}{dt} = K + a\left(k\rho + \frac{K}{\rho}\right)\cos(v - \theta) - a\sin(v - \theta)\frac{d\rho}{dt}. \quad (53)$$

RECTILINEAR MOTION

The relative motion in some double stars is apparently rectilinear and it is desirable to have criteria which shall enable us to decide whether this is due to the fact that the orbit is a very elongated ellipse, or to the fact that the two stars are unrelated and are changing their relative positions by reason of the difference in their proper motions. One excellent test, which has been applied by Lewis to many of the Struve stars, is that if the stars are unrelated the apparent motion of the companion referred to the primary will be uniform whatever the angular separation of the stars; but if they form a physical system, it will increase as the angular distance diminishes.

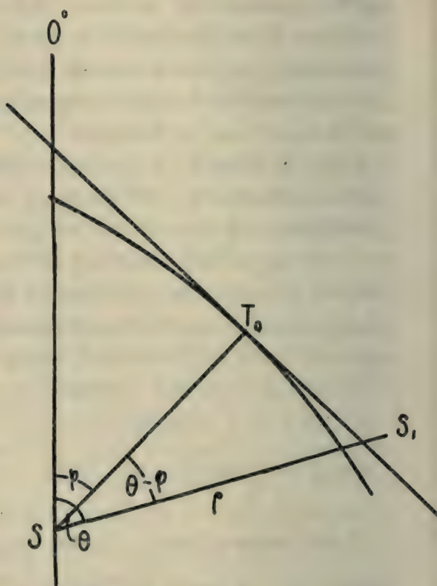

FIGURE 6. Rectilinear Motion

A more rigorous test is the one applied, for example, by Schlesinger and Alter [11] to the motion of 61 *Cygni*. If the motion is uniform and in a straight line, the position angles and distances of the companion referred to the primary may be represented by the equations

$$\begin{aligned} \rho^2 &= a^2 + (t - T)^2 m^2 \\ \tan(\theta - \phi) &= \frac{m}{a}(t - T) \end{aligned} \qquad (54)$$

in which a is the perpendicular distance from the primary, considered as fixed, to the path of the companion; ϕ is the position angle of this perpendicular; T, the time when the companion was at the foot of the perpendicular, and m, the annual relative rectilinear motion of the companion. Approximate values for these four quantities may be obtained from a plot of the observations and residuals may then be formed

[11] *Publications Allegheny Observatory* 2, 13, 1910.

by comparing the positions computed from the formulae with the observations. If these residuals exhibit no systematic character, rectilinear motion may be assumed; if they show a systematic course a closer examination is in order to decide whether this is due to chance or to orbital motion. In the latter case, the indicated curve must be concave to the primary and the systematic run of the residuals should be quite uniform. In any event, a least squares solution may be made to obtain more precise values for the quantities a, ϕ, T and m. For this purpose, differentiate equations (54) and introduce the values $\sin(\theta-\phi)=\frac{m(t-T)}{\rho}$, $\cos(\theta-\phi)=\frac{a}{\rho}$ (see Fig. 6); we thus obtain the equations of condition in the form given by Schlesinger and Alter:

$$\begin{aligned} -\cos(\theta-\phi_\circ)\Delta a - \sin(\theta-\phi_\circ)(t-T_\circ)\Delta m& \\ + \sin(\theta-\phi_\circ)m_\circ\Delta T + \Delta\rho &= v_\rho \qquad (55)\\ +\sin(\theta-\phi_\circ)\Delta a - \cos(\theta-\phi_\circ)(t-T_\circ)\Delta m& \\ + \cos(\theta-\psi_\circ)m_\circ\Delta T - \rho\Delta\psi + \rho\Delta\theta &= v_\theta \end{aligned}$$

n which the subscript $_\circ$ indicates the preliminary values of the elements, $\Delta\rho$ and $\Delta\theta$ the deviations from the approximate straight line and v_ρ and v_θ the residuals from the definitive values of the elements.

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CHAPTER V

THE RADIAL VELOCITY OF A STAR

By J. H. Moore

The observations treated in the preceding chapters concern only that part of the star's actual motion in space, which appears as change of position in a plane perpendicular to the line joining the observer and star. Of the component directed along the 'line of sight', called the star's 'radial motion', the telescope alone gives no indication. In fact, the possibility of detecting radial motion was recognized less than seventy-five years ago, and the methods of its measurement belong distinctly to another and newer branch of astronomy, known as astrophysics. Moreover, observations of the rate of change of position of a star on the celestial sphere can be translated into linear units, such as kilometers per second, only if the star's parallax is known, while measures of radial velocity by the method to be described, are expressed directly in kilometers per second and are independent of the star's distance.

The determination of the radial velocity of a light source, such as a star, is made possible by two well-known properties of light; namely, that it is propagated as a wave motion, and with a definite and finite velocity. We are not concerned with the properties of the hypothetical medium, called the ether, in which these waves move, nor with the nature of the disturbance in the ether, whether it be mechanical or electromagnetic. For our purpose it is sufficient to know that in this medium, or in interstellar space, the velocity of light is about 299,860 kilometers per second, and that the well-known laws of wave motion hold for light waves.

In 1842, Christian Doppler called attention to an effect upon the apparent length of a wave which should result from a relative motion of the source of the waves and the observer. This result was independently reached and further developed, especially with reference to light waves, some six years later by the great French physicist, Fizeau. According to the Doppler-Fizeau principle, when the relative motion of the light source and the observer is such, that the distance between the two is increasing or decreasing, the length of the waves received by the observer will be respectively longer or shorter than the normal length of these waves.

It is readily shown that the change in wave-length is directly proportional to the normal length of the wave and to the ratio of the relative velocity of source and observer to the velocity of propagation of the waves. Moreover, the change is the same whether the source or observer, or both are moving, providing their velocities are small in comparison with that of the waves. In the case of light waves, and for the celestial objects with which we have to deal, this condition is always fulfilled.

Let us denote by v the relative radial velocity in kilometers per second of a star and observer, where v is considered positive when the distance between the two is increasing and negative when this distance is decreasing. Call λ' the wave length of a monochromatic ray reaching the observer, whose normal wave-length, as emitted by the star is λ.

Then from the Doppler-Fizeau principle, $\lambda' - \lambda : \lambda :: v :$ 299,860;[1] or $\lambda' - \lambda = \lambda v / 299860$ (if v is $+$, λ' is greater than λ), or, writing $\Delta\lambda$ for the change in wave length $(\lambda' - \lambda)$, we have for the relative radial velocity of star and observer

$$v = \frac{299860 \, \Delta \lambda}{\lambda}. \qquad (1)$$

The determination of the radial velocity of a star rests then upon a knowledge of the velocity of light and of the wave-lengths of certain definite rays emitted by a source at rest, and the measurement of the apparent wave-lengths of those same rays received from a star. In short, the problem reduces to one of measuring $\Delta\lambda$ with the greatest possible precision. For this purpose the micrometer with which we have become familiar,

[1] The velocity of light, in kilometers per second.

is replaced by the spectroscope. This wonderful instrument originating in the physical laboratory has developed a whole new science, spectroscopy, with an extensive and technical literature of its own. In this chapter we shall only call attention to some of the elementary principles of spectroscopic analysis and give a very brief survey of the spectrographic method as applied to the determination of stellar radial velocities. The student who wishes to pursue the subject further, will find a list of references to extended treatment of the various topics at the end of this chapter.

Since stellar light sources are very faint in comparison with those available in the laboratory, it is necessary to employ for this special problem the spectroscope which is the least wasteful of light. For this reason the prism-spectroscope is the only one of the various laboratory forms which is at present generally applicable to stellar spectroscopy and we, therefore, limit our discussion to this particular type.

The essential parts of a laboratory spectroscope and their principal functions are briefly as follows: Light from the source to be studied is brought to a focus by a condensing lens on the narrow slit of the spectroscope. After passing through the slit, the rays are rendered parallel by an achromatic converging lens, called the 'collimator' lens. The rays then strike a glass prism, placed with its apex parallel to the length of the slit, by which they are bent from their original direction. It is here that we obtain the separation of the rays, since the amount by which each ray is deviated by the prism is a function of its wave-length. The direction of the long red waves is changed the least, while the shorter violet ones suffer the greatest deviation. After each set of rays is collected and brought to its corresponding focus by a second achromatic converging lens, we shall have an orderly array of images of the slit, each image formed by light of a definite wave-length. Such a series of images is called a 'spectrum' of the source. The spectrum may be viewed with an ordinary eye-piece, or the second lens may be used as a camera lens, and the spectrum be recorded on a photographic plate placed in its focal plane. In all stellar work the spectroscope is employed photographically, in which case it is called a 'spectrograph', and the photograph obtained with it is a 'spectrogram'.

If the slit is made extremely narrow there will be very little overlapping of the images and the spectrum is then said to be 'pure'. It can be shown that the purest spectrum is obtained when the incident rays fall upon the prism at such an angle that they will be least deviated from their original direction by the prism. It is well known that this position of *minimum deviation* is also the one of maximum light transmission by the prism; and it has the further advantage that any accidental displacement of the prism produces the minimum displacement of the spectrum line. The prism or prisms of stellar spectrographs are therefore always set at the angle of minimum deviation for the approximate center of the region of spectrum to be studied.

Attention was called in an earlier chapter to two factors which define the optical efficiency of a telescope for the separation of close double stars, *viz*: (*a*) the resolving power of the objective, (*b*) the magnification or linear distance between the two images at the focus of the objective. These same factors form a convenient basis for the comparison of the resolving powers of two spectrographs. Here, however, we are concerned with the separation of two images of the slit formed by light of different wave-lengths. The resolving power of a spectrograph is, therefore, defined as *the minimum difference of wave-length between two lines for which the lines will just be separated.* It is a function of the width of slit, the wave-length, and the difference between the maximum and minimum lengths of path of the rays in the prism. The magnification, called 'the linear dispersion' of the spectrograph, is expressed, as the number of wave-length units per unit length of spectrum and depends upon the wave-length of the ray, the optical constants for the prism system and the focal length of the camera lens.

When the slit of a spectroscope is illuminated by the light from an incandescent solid, such as the filament of an incandescent lamp, or from an incandescent gas under high pressure, the spectrum consists of an unbroken band of color; that is, a *continuous spectrum.* An incandescent gas or vapor under low pressure gives a spectrum consisting of isolated bright line images of the slit—a *bright line spectrum*—the bright lines indicating that radiations of certain definite wave-length are emitted by the gas. Each chemical element, in the gaseous state, when rendered luminous in the electric arc, electric spark, flame, or vacuum tube, gives its own set of bright lines, which are characteristic of this element alone and whose wave-lengths remain constant for a source at rest under the same conditions of temperature, pressure, etc.

An incandescent gas has the property not only of radiating light of certain definite wave-lengths, but also of absorbing, from white light passing through it, the rays of precisely those same wave-lengths. If the temperature of the incandescent gas is lower than that of the source behind it, the continuous spectrum will be crossed by relatively dark lines, whose positions agree exactly with the bright line spectrum characteristic of the gas. This relation existing between the emission and absorption of a gas is known as Kirchoff's law, and the type of spectrum described is termed an *absorption spectrum.*

The three principles just stated obviously lead to a simple and direct method of analyzing the chemical constituents of gaseous light sources, and of furnishing information as to their physical conditions. A certain class of nebulae, for example, give bright line spectra, indicating that they are masses of luminous and extremely rarefied gases. Most of the stars, including our own sun, give absorption spectra, showing that the light emitted by a central glowing core has passed through a surrounding atmosphere of cooler vapors. The presence of most of the known chemical elements in the atmospheres of the Sun and stars has been recognized from the lines in the spectra of these objects. In addition, there occur in them many lines, which have not yet been identified with those of any known element.

The length of the light wave for each line is such a minute fraction of a millimeter that spectroscopists have adopted as the unit of wave-length, the 'Ångström', equal to one ten-millionth of a millimeter, for which *A* is the symbol. Thus the wave-length of the hydrogen radiation in the violet is 0.0004340 mm. or 4340A.

Measures of the wave-lengths of the lines in a star's spectrum are readily effected with the prism spectrograph, by a comparison of the positions of the stellar lines with those from a source the wave-lengths of whose lines are known. To accomplish this the light from a suitable source (for example the iron arc) is made to pass over very nearly the same path in the spectrograph as the star's light travels, and the spectrum of this source, termed the *comparison spectrum*, is recorded on each side of the star spectrum.

When the spectra of a number of stars are examined, it is found that they exhibit a great variety in the number and character of their lines. From an examination of several hundred stars by means of a visual spectroscope, Secchi about 1866–1867 was able to arrange their spectra under four types. While exhibiting very well the most prominent characteristics of stellar spectra, his system is insufficient for portraying the finer gradations, which the photographic method has brought to light. The classification now in general use among astrophysicists, was formulated by Professor Pickering, Miss Maury, and Miss Cannon from the very extensive photographic survey of stellar spectra made at the Harvard College Observatory and at the Harvard station at Arequipa, Peru. It is based upon the observed fact that certain groups of lines have a common behavior. They make their appearance and increase or decrease in intensity at the same time, so that a more or less orderly sequence of development from one type of spectrum to another is indicated.

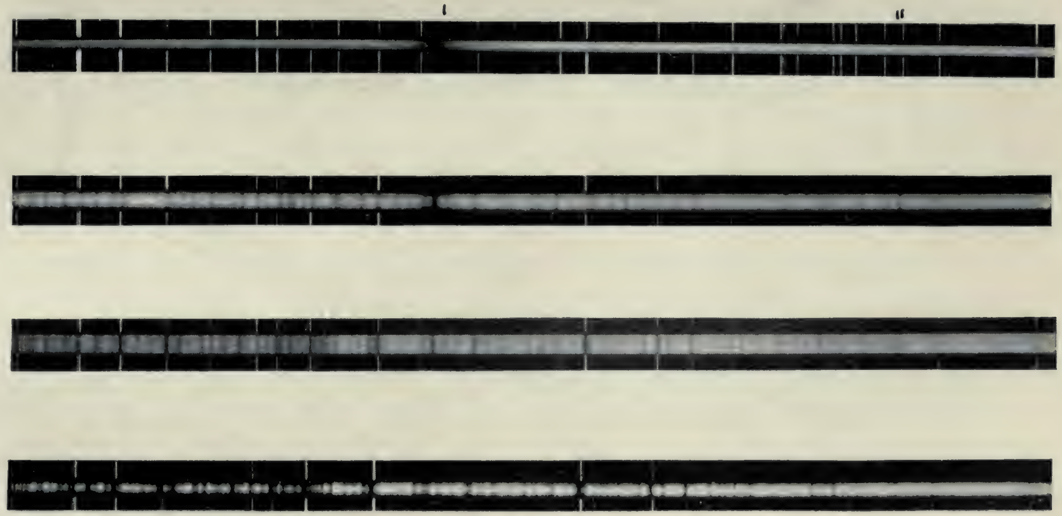
A very condensed outline of this system of classification will serve to indicate its chief features. Its main divisions, arranged in the supposed order of development, which is that of the more generally accepted order of stellar evolution, are represented by the capital letters *P*, *O*, *B*, *A*, *F*, *G*, *K*, *M*, (*R*, *N*). Sub-groups are indicated by small letters or numbers on the scale of ten. To class *P* are assigned all bright line nebulae, while the other classes refer to stellar spectra.

Spectra of Class *O*, in the five sub-divisions *Oa* to *Oe*, contain a group of bright bands of unknown origin, and also the first and second series of hydrogen lines, which are bright in *Oa-Oc*, and dark in *Od* and *Oe*. Toward the end of the class some of the so-called 'Orion lines', or dark lines due chiefly to helium, nitrogen, silicon, magnesium and carbon, begin to appear. In Class *Oe5*, intermediate between *Oe* and *B*, the bright bands have disappeared. The secondary series of hydrogen vanishes early in Class *B*, while the primary series increases in intensity throughout the ten subdivisions, *Bo*, *B1*, etc. Near the middle of the group the Orion lines begin to disappear, and toward the end, in *B8* and *B9*, some of the metallic lines are faintly visible. In the Classes *Ao* and *A2*, the primary hydrogen series reach their maximum intensity and decrease in the other two subdivisions, *A3* and *A5*. The calcium lines, *H* and *K*, and those due to the metals increase in prominence through this class and the four subdivisions of Class *F*. In Class *G*, which includes stars whose spectra closely resemble that of the Sun, the *H* and *K* lines and a band designated by *g* are the most conspicuous features, whereas the hydrogen lines are scarcely more prominent than many of the metallic lines. Classes *G5*, *Ko*, and *K2*, represent spectra of a type a little more advanced than that of the Sun. Class *K* is further characterized by a decrease in intensity of the continuous spectrum in the violet and blue. This becomes quite marked in Classes *K5* and *Ma*, *Mb*, and *Md*. The three divisions of Class *M* are further distinguished by absorption bands of titanium oxide, which first make their appearance in *K5*. Stars of Class *Md* show in addition bright hydrogen lines. To Class *N* belong stars whose spectrum of metallic lines is similar to that of *M*, but which are particularly characterized by a banded spectrum ascribed to carbon absorption. Class *R* includes stars whose spectra are similar to those of Class *N*, except that they are relatively more intense in the violet. These two classes we have placed in brackets in the arrangement according to development, since some uncertainty exists as to the place they should occupy in such a scheme. Stars of Classes *O*, *B*, and *A* are bluish white in color.

F, *G* and *K* stars are yellow. Those of Class *M* are red or orange, while the *N* stars are a deep red.

In Plate III are reproduced four stellar spectrograms secured with the three-prism spectrograph of the D. O. Mills Expedition, at Santiago, Chile, which illustrate the different appearance of the spectra in the blue-violet region of Classes *B8*, *F*, *G*, and *K5*. On all of the spectrograms the bright line spectrum of the iron arc was photographed above and below the star spectrum. The spectrum of υ_4 *Eridani* (Figure *a*) of Class *B8*, shows only the hydrogen line $H\gamma$ (4340.634*A*) and the magnesium line (4481.400*A*), as the very faint metallic lines, some of which appear on the original negative, are lost in the process of reproduction. This star is a spectroscopic binary, and the spectra of both stars are visible, so that each of the two lines mentioned above is double. The strengthening of the metallic lines and the decrease in intensity of $\mathrm{H}\gamma$ is shown in the spectrum of α *Carinae* of Class *F* (Fig. *b*), while in the solar spectrum (Fig. *c*), of Class *G*, and in that of α_2 *Centauri* (Fig. *d*), of Class *K5*, a further decrease in $H\gamma$, the disappearance of 4481*A* and a considerable increase in the number and strength of the absorption lines of other elements are noticeable.

The four spectrograms illustrate also the displacement of the lines in star spectra as effects of motion in the line of sight. The iron lines in the solar spectrum are practically coincident with the corresponding lines of the iron arc, since the relative radial velocity of the Sun and the observer is very small. The iron lines in the spectrum of α *Carinae* are clearly displaced from their normal positions, as given by the lines of the comparison spectrum. This displacement is toward the red end of the spectrum, and corresponds, therefore, to an increase in the wave-lengths of the star lines. Interpreted on the Doppler-Fizeau principle, this change is produced by a recession of the star with respect to the Earth at the rate of $+25.1$ km. per second. In the case of α_2 *Centauri*, the displacement of the lines is toward the violet and corresponds to a velocity of approach of -41.3 km. per second. As an example of the Doppler-Fizeau effect, the spectrogram of the spectroscopic



(a)

(b)

(c)

(d)

PLATE III. Spectra in the Blue-Violet Region of (a) υ_4 Eridani, (b) α Carinae, (c) the Sun, (d) α_2 Centauri

binary ν_4 *Eridani*, is perhaps the most striking. The two stars revolve about their common center of mass in a period of 5.01 days, as shown by an extended series of plates similar to this one. Due to their orbital motion, the velocity of each star in the line of sight is continually changing, giving rise to a continuous variation in the separation of the lines of the two spectra. The spectrogram reproduced here, was taken at the time of maximum velocity of approach of one, and the corresponding velocity of recession of the other component. It shows, therefore, the maximum separation of the lines of the two spectra. The relative radial velocity of the two stars was 126 km. per second. Obviously, the lines of the two spectra will be coincident when the motion of the two components is across the line of sight, which occurs at intervals of 2.5 days.

It is well known that the wave-lengths of spectral lines are affected by other causes than that arising from radial motion of the source. For example, it is found that an increase in pressure of the emitting or absorbing vapor will in general shift the lines toward the red. This effect, even with considerable pressures, is small and is moreover not the same for all lines. Of the many conditions which displace spectrum lines, radial motion is the only one of which measures of stellar spectra have furnished reliable evidence.

Displacements of the stellar lines with reference to those of the comparison spectrum, may arise wholly or in part from causes which are purely instrumental. Thus, if the star light and the artificial light do not pass over equivalent paths in the spectrograph, or if a change in the relative positions of the parts of the instrument occurs between the times of photographing the stellar and the reference spectra, a relative displacement of the lines of the two spectra will result. The first-named source of error is an optical condition, to be met for all spectroscopic measures, that is easily satisfied. With the conditions of a fixed mounting and approximately constant temperature, under which the spectrograph is used in the laboratory, the second source of error need not be considered. When, however, the spectrograph is applied to stellar observation, it is necessary, in order to avoid undue loss of light, to mount it on a moving telescope, and hence to subject the instrument to the varying component of gravity and the changing temperature of a well-ventilated dome. The spectrograph must be so designed and constructed that it will be free from appreciable differential flexure in any two positions of the telescope, and provision must be made against the disturbing effects of temperature changes in the prisms and of the metal parts of the instrument. Further, in addition to the obvious requirement that the prisms and lenses shall give good definition, they must be so chosen and arranged as to give satisfactory resolving power with efficiency in light transmission.

The earlier determinations of stellar radial velocities were made entirely by the visual method. Although made by such skilled observers as Huggins, Vogel, and others, the errors of observation, except for a very few of the brightest stars, often exceeded the quantities to be measured. After the introduction of the photographic method of studying stellar spectra, Vogel and Scheiner, at Potsdam, and later Belopolsky, at Pulkowa, were able to measure the radial velocities of the brightest stars with an average probable error of ±2.6 km. per second. In 1895–1896 the problem was attacked by Campbell, who employed a specially designed stellar spectrograph—the Mills Spectrograph—in conjunction with the thirty-six-inch refractor of the Lick Observatory. For the brighter stars, the probable error of his measures was about ±0.5 km. and for bright stars whose spectra contain the best lines, the probable error was reduced to ±0.25 km. Many improvements in stellar spectrographs have, of course, been made in the succeeding twenty-one years, but the standard of precision set by his measures represents that attained today for the same stars. The advances which have been made in this time relate more to the increased accuracy of the results for fainter stars.

Now this remarkable advance in the precision of the measures made by Campbell was due not to the use of a great telescope but to the fact that his spectrograph was designed in accordance with the important requirements mentioned above—excellence of definition and maximum light transmission,

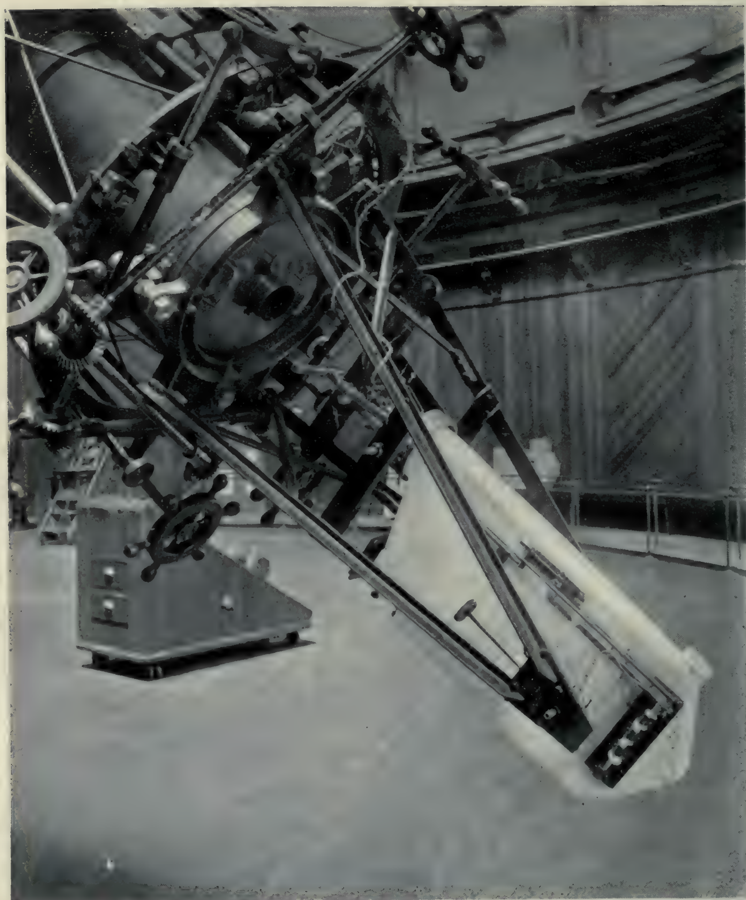


PLATE IV. The Mills Spectrograph of the Lick Observatory rigidity and temperature control of the spectrograph—and to improved methods of measuring and reducing the spectrograms.

In order to understand more clearly the manner in which the optical and mechanical requirements are met in practice, a detailed description will be given of a modern spectrograph which was designed to have maximum efficiency for the particular problem of determining stellar radial velocities. A view of the new Mills spectrograph attached to the thirty-six-inch refractor of the Lick Observatory is presented in Plate IV. The essential parts of this instrument are the same as those described for the simple laboratory spectrograph; namely, the slit, collimator lens, prism and camera lens, except that here *three* 60° prisms of flint glass are employed. The prisms, set at minimum deviation for 4500 A, produce a deviation of this ray of 176°. A rectangular box constructed of saw-steel plates, to which are connected respectively the slit mechanism, the prism box, and the plate holder, by three light steel castings, forms the main body of the spectrograph. In the casting to which the prism box is attached are mounted the collimator and camera lenses, both of which are achromatic for the region of 4500 A. The spectrograph has an entirely new form of support, designed by Campbell, to incorporate the suggestion made by Wright, that such an instrument should be supported *near its two ends*, like a bridge truss or beam, in order to give minimum flexure. The support is a frame work of T-bars extending down from the telescope, the form and arrangement of which is such as to hold the instrument rigidly in the line of collimation of the large telescope. The lower support is a bar passing through a rectangular opening in the casting carrying the prism box. This bar is pivoted at the center of the casting and connected at its two ends to the supporting frame. The upper support consists of a cylindrical ring firmly attached to the frame work. In this cylinder fits a spherical flange of the spectrograph casting, the two forming a universal joint. Any strains originating in the supporting frame cannot, with this form of mounting, be communicated to the spectrograph. Careful tests of this instrument and of the spectrograph of the D. O. Mills Expedition to Chile, which has the same form of mounting, show that the effects of differential flexure have been eliminated. This method of support permits, further, of a very convenient mode of moving the spectrograph as a whole in order to bring the slit into the focal plane of the large telescope, since it is only necessary to provide sliding connections on the frame, for the lower support.

Nearly all modern stellar spectrographs are provided with reflecting slit plates inclined at a small angle to the collimation axis, which enable the observer to view the star image directly on the slit. This is accomplished through the aid of a total-reflection prism, placed above the slit and outside of the cone of rays from the telescope objective, which receives the light from the slit and sends it to the guiding eyepiece. By placing the slit parallel to the celestial equator, small errors of the driving clock cause the star image to move along the slit, which is desirable in order to obtain width of spectrum. *Constant and careful guiding is necessary to insure that the star's image be kept exactly on the slit and that its motion along the slit be such as to give a uniform exposure.*

With a prism spectrograph and a straight slit the spectrum lines are curved. The amount of the curvature depends upon the optical constants of the instrument and the wave-length of the line. This source of trouble in measuring the spectrograms may be eliminated for a short range of spectrum by employing a slit of the proper curvature to make the spectrum lines straight. Both three-prism instruments referred to above are provided with curved slits.

As a source for the comparison spectrum, it is necessary to select one giving a number of well-distributed lines in the part of the spectrum to be studied. For example, for the new Mills spectrograph in which the region 4400 A to 4600 A is utilized, the spark spectrum of titanium is used. In the southern instrument, arranged for the region 4200 A–4500 A, the comparison source is the iron arc.

In order to eliminate the effects of any possible change in the instrument during an exposure on the star, several impressions of the comparison spectrum are made at regular intervals. This is accomplished very conveniently and without danger of changing the adjustment of the comparison apparatus by a simple device due to Wright. Two small total-reflection prisms are placed just above the slit, so that their adjoining edges define the length of the slit. Two light sources are then so arranged that the beam of each is brought to a focus on the slit by a small condensing lens after total reflection in its respective prism.

The optical parts of the spectrograph should, of course, be mounted so that they cannot move, but care must be taken that they are not cramped. This caution is especially pertinent with regard to the large prisms. In the Mills spectrographs the prisms rest upon hard-rubber blocks and are firmly clamped to one of the side plates of the prism box by light steel springs which press against their upper surface. Small hard-rubber stops prevent lateral motion of the prisms.

In order to prevent the effects of changing temperature, the principal parts of the spectrograph are surrounded by a light wooden box, lined with felt. Over the felt surface are strung a number of turns of resistance wire. The regulation of the heating current is effected by means of a very sensitive mercury-in-glass thermostat by which the temperature inside of the prism box is held constant during the night's work to within a few hundredths of a degree Centigrade.

The function of the telescope objective, for observations of stellar spectra, is that of a condensing lens and the brightness of the point image in the focal plane is directly proportional to the area of the lens and its transmission factor. If we had perfect 'seeing' we should receive in the slit of the spectrograph, with the widths generally employed, about ninety per cent. of the light in the star image. Due to atmospheric disturbances the image of a star under average conditions of seeing, is a circular 'tremor' disc whose diameter is four or five times the width of the slit, so that the brightness of the spectrum is not proportional to the *area* of the objective but more nearly to its *diameter*. For example, the relative intensities of stellar spectra obtained with the same spectrograph respectively upon the thirty-six-inch and twelve-inch refractors of the Lick Observatory would be (allowing for the difference of transmission of the two), about as two to one, since, for the photographic rays, the loss of light is for the former about fifty per cent. and for the latter about twenty-five per cent. When a visual refractor is used for spectroscopic work, it is necessary to render it achromatic for the photographic rays. This is accomplished for the thirty-six-inch refractor by a correcting lens of 2.5 inches aperture placed one meter inside the visual focus of the telescope. This lens introduces an additional loss of light of fully ten per cent.

Since a silver-on-glass mirror has, under the best conditions, a high reflecting power, and since it is also free from chromatic aberration, it would seem that the reflector should be the more efficient telescope to use in connection with a stellar spectrograph. The reflector, however, possesses its own disadvantages, one of which is that it is very sensitive to changes of temperature. Our experience with the thirty-six-inch refractor at Mount Hamilton and the thirty-seven and one-half inch reflector in Chile, when used with high dispersion spectrographs, indicates that the relative light efficiency of the two is about equal in the region of $H\gamma$. For apertures up to thirty-six inches one is inclined to favor the refractor for high dispersion work, while for low dispersion, where considerable extent of spectrum is desired, the reflector is, of course, preferable.

The focal lengths of both refracting and reflecting telescopes vary with change in temperature of the lens or mirror. It is, therefore, necessary before beginning the night's work, and with the reflector frequently during the night, to bring the slit into the focal plane of the telescope, which as noted above, is effected by moving the spectrograph as a whole in the line of collimation of the instrument.

It is well known that all high dispersion spectrographs are very wasteful of light, though to what extent is perhaps not always appreciated. When stellar spectrographs of three-prism dispersion are used in conjunction with large refractors or reflectors the combined instrument delivers to the photographic plate probably less than two per cent. of the light incident upon the telescope objective. Half of the light is lost, as we have noted, before it reaches the slit. The remaining losses occur at the slit, in the prisms and in the collimator and camera lenses of the spectrograph. In order to avoid unnecessary losses of light, the obvious conditions must be satisfied, that the angular apertures of the collimator lens and object glass are the same, and that the prisms and camera lens are of sufficient aperture to admit the full beam from the collimator. The most serious losses occur at the narrow slit and in the prism train. Indeed, one of the most important factors in the design of stellar spectrographs, for maximum light efficiency, is the proper balancing of these two conflicting elements, the transmission at the slit and the transmission of the prisms. Thus, in the new Mills spectrograph, by using a collimator of slightly greater focal length than the present one (28.5 inches) with corresponding increase in aperture of the lens and prisms, a wider slit could be employed and still maintain the present purity of spectrum. After allowance is made for the increased absorption of the prism train, there would remain a small gain in light transmitted. Although theoretically possible, this gain would probably be more than offset by the inferior definition of the larger prisms and the added difficulty of eliminating flexure. It is necessary here, as at so many points in the spectrograph, to sacrifice a little in order to gain more elsewhere. In fact, the most efficient design of spectrograph may be described as the one in which the wisest compromises have been made between the various conflicting interests.

The decision as to the resolution and dispersion to be employed is governed by several considerations: the type of stellar spectrum to be studied, the size of the telescope at one's disposal, and the brightness of the source whose spectrum can be photographed with reasonable exposure times. With the spectrograph here described two lines in the region of 4500 A whose wave-lengths differ 0.2 A are resolved, while the linear dispersion for 4500 A is 1 mm. = 11 A. In order to obtain a spectrogram of suitable density of a star whose photographic magnitude is 5.0, an exposure time of an hour and a half is required. For stars of photographic magnitudes 6.0 to 6.5 the width of slit is increased, thus sacrificing to some degree the purity of spectrum, but not enough to interfere seriously with the accuracy of the measures. In the case of early type stars whose spectra contain single lines, the question of resolution is not important, and where these lines are also broad, it is preferable to employ lower dispersion. The adjustments of the various parts of the spectrograph call for continual attention. It is necessary that the instrument be placed with its axis of collimation accurately in that of the large telescope and frequent tests should be made to be sure that it remains so. The comparison source must be adjusted so that its light follows very nearly the same path as the star light in the spectrograph. Care must be exercised at every point in the process of obtaining and measuring the spectrogram.

THE MEASUREMENT AND REDUCTION OF SPECTROGRAMS

For the measurement of spectrograms any one of the usual forms of laboratory measuring microscopes will suffice. This is merely a microscope on the stand of which is mounted a carriage, movable by an accurate micrometer screw, in a direction at right angles to that of the microscope axis.

In order to fix ideas, we shall assume that it is required to measure and reduce the spectrogram of α_2 *Centauri*, the positive of which is reproduced in Plate III. The spectrogram is first clamped on the carriage of the microscope, and the usual adjustments of focus and alignment of the plate are made. Great care should be taken that the illumination of the field of the microscope is uniform. Beginning with the comparison line 4250 A, settings are made continuously along the plate on good star lines and comparison lines as they chance to occur. The plate is then reversed and the settings are repeated. It has been shown by several investigators that the effects of errors due to personal equation are practically eliminated by taking the mean of the measures in the two positions. In the reversal of the plate the spectrum is also inverted, which may so change the appearance of the lines as to interfere with the elimination of personal equation. Especially is this true if the lines are curved. The effects of accidental errors in setting are reduced by employing a number of lines.

The accompanying table contains the data of the measure and reduction of this plate. Column I gives the wave-lengths of the lines of the iron comparison and the normal wave-lengths of the star lines, taken from Rowland's 'Preliminary Table of Solar Wave-lengths'. In columns IV and V are recorded respectively the settings on the comparison and star lines (in revolutions of the micrometer screw). The displacements of the iron lines in the star are evidently given directly in amount and sign by the difference, star minus comparison and these are entered at once in column VII (Displ.). We cannot enter the displacements for the other star lines until the normal positions of these lines have been obtained from those of the iron comparison, by interpolation. This is effected in the following manner:

A smooth curve drawn by plotting, for the comparison lines, the reading on each line and its corresponding wave-length, respectively as ordinates and abscissae, will evidently represent for this spectrogram the relation existing between wave-length and micrometer readings. From this curve—called a 'dispersion curve'—either the zero readings or the observed wave-length of the stellar lines could be obtained. This curve was found by Cornu and later by Hartmann to be nearly of the form of an equilateral hyperbola so that it is approximately represented by the equation

$$x - x_{\circ} = \frac{c}{\lambda - \lambda_{\circ}}, \qquad (2)$$

where x is the micrometer reading on a line whose wave-length is λ and λ_o, x_o, and c are constants. Since it is not practicable to plot the dispersion curve, the Cornu-Hartmann formula furnishes a very convenient means of obtaining it. The values of the three constants are determined from three equations formed by substituting the micrometer readings and wave-lengths of three lines, selected, one at each end of the region of spectrum and the other near the middle. Micrometer read-

* α_2 Centauri ft. *Plate No. 3791 III α 14^h 32.8^m
Date 1911 Feb. 27

λ	*Table*	*Co–Ta*	*Comp.*	*	*Sup'd*	*Displ.*	rV_s	v_s
4250.287	54.886	0	54.886	54.758		–.128	319	–40.8
4250.945	55.031	3	55.034	54.909		–.125	320	–40.0
4282.565	61.819	13	61.832	61.710		–.122	335	–40.9
4283.169	61.944			61.831	958	–.127	335	–42.5
4294.301	64.250	16	64.266	64.140		–.126	338	–42.6
4299.410	65.295	20	65.315	65.190		–.125	340	–42.5
4313.034	68.039			67.944	061	–.117	349	–40.8
4313.797	68.190			68.090	212	–.122	349	–42.6
4318.817	69.185			69.105	220	–.115	352	–40.5
4325.152	70.431			70.355	469	–.114	356	–40.6
4325.939	70.584	40	70.624	70.502		–.122	356	–43.4
4328.080	71.001			70.928	041	–.113	357	–40.3
4337.216	72.767	43	72.810	72.692		–.118	360	–42.5
4340.634	73.421			73.350	467	–.117	362	–42.4
4359.784	77.027			76.970	082	–.112	372	–41.7
4369.941	78.896			78.844	957	–.113	376	–42.5
4376.107	80.018			79.972	083	–.111	378	–42.0
4379.396	80.612			80.571	680	–.109	380	–41.4
4383.720	81.388	70	81.458	81.352		–.106	382	–40.5
4399.935	84.257			84.228	337	–.109	390	–42.5
4404.927	85.126	86	85.212	85.105		–.107	392	–41.9
4406.810	85.453			85.432	539	–.107	394	–42.2
4415.293	86.913	93	87.006	86.898		–.108	397	–42.9
4425.608	88.664			88.662	759	–.097	402	–39.0
4428.711	89.198			89.194	296	–.102	404	–41.2
4430.785	89.536			89.535	636	–.101	404	–40.8
4435.129	90.270			90.270	372	–.102	406	–41.4
4435.851	90.380			90.378	482	–.104	406	–42.2
4442.510	91.482	108	91.590	91.491		–.099	411	–40.7
4443.976	91.724			91.732	831	–.099	412	–40.8
4447.892	92.365			92.375	473	–.098	413	–40.5
4459.301	94.216	114	94.330	94.238		–.092	417	–38.4
4476.185	96.906	127	97.033	96.940		–.093	426	–39.6
4482.379	97.872	131	98.003	97.905		–.098	428	–41.9
4494.738	99.782	138	99.920	99.820		–.100	434	–43.4

35)1449.9
–41.43
Scale= + 0.13
va= +21.82
vd= – 0.07
Observed *V* –19.55 km.

ings of all other comparison and star lines are then computed from the formula. The departure of this computed curve from the true dispersion curve is furnished by a plot of the differences between the observed and calculated readings of the comparison lines. The computed normal positions of the star lines are then corrected for the difference between the computed and observed dispersion curve. The decimal portions of the results would be entered in column VI (Sup'd).[2]

As before, the difference, star line minus zero line, gives the displacement in revolutions of the screw. In order to express this as $\triangle\lambda$, that is in units of wave-length, it is necessary to know r, the number of angstrom units in one revolution of the screw. The value of r for any point in the spectrum is evidently the slope of the dispersion curve at that point, and is equal to $\frac{\lambda - \lambda_0}{x - x_0}$. Finally, in accordance with the relation deduced earlier, v the observed radial velocity is obtained by multiplying $\triangle\lambda$ for each line by its corresponding factor $V_s = \frac{299860}{\lambda}$.

Each spectrogram may be reduced in the manner outlined above, and some observers prefer to follow this method rigorously for each stellar spectrogram. When this is done the process is simplified by carrying through the computation in wave-lengths, so that the displacement is expressed at once in angstroms.

Since for the same spectrograph the form of the dispersion curve differs but slightly for different temperatures, a simple and practical method of reduction is offered by the following procedure: A standard dispersion curve is computed once for all, according to the method described above, from measures of a solar spectrogram. With the aid of this all other spectrograms taken with the spectrograph may be quickly and easily reduced. It is convenient to put this standard curve in the form of a dispersion table in which are entered the normal wave-lengths of the comparison and stellar lines used

[2] The figures actually entered in this column in the example were obtained by a different method of reduction which is explained in the paragraphs following.

for stars of different spectral classes, and the micrometer readings corresponding to these wave-lengths. In this standard table are given also the values of rV_s for each line. Columns I and II and VIII, in the example, are taken from such a table.

It is now only necessary to reduce the readings of the standard table to the dispersion of the plate, by plotting the differences between the observed and table readings of the comparison lines (recorded in Column III in the example). From this curve the difference to be applied to the table reading for each star line is read off. In the sixth column are given the new table readings (for zero velocity) after this difference has been applied. When there are comparison lines corresponding to star lines some observers follow rigorously the process outlined, while others (as in the present example) take the difference between the readings of the two as the displacements. The last three columns contain, respectively, the displacements (* minus Comp. or Sup'd), the factor rV_s, and the products of these two values, which are the relative radial velocities of star and observer as supplied by the lines measured. The mean of the measures for forty lines gives as the observed radial velocity -41.43 km./sec. It will be noticed that the dispersion of the star plate is about three-tenths of one per cent. greater than that of the standard table, and consequently the factor r (computed for the table) is too large, and the numerical value of this velocity must be reduced by this amount. This is allowed for, in the example, as scale correction. In practice, it is convenient to have several standard tables corresponding to the dispersion of the spectrograph at different temperatures. The one whose dispersion is nearest that of the star plate is selected for use. Experience has shown that the results obtained by the very simple method just described are of the same accuracy as those derived by the longer process of computing a dispersion curve for each plate.

If the spectrograph is not provided with a curved slit it is necessary to introduce a correction for the curvature of the lines. This correction may be computed from Ditscheiner's formula[3] or determined empirically from lines on a spectrogram of the Sun, on the assumption that the curve of each line is a parabola. The better method is to eliminate the source of this correction by the use of a curved slit.

The observed radial velocity of a star is made up of the star's velocity, V, with reference to the solar system, and the velocity of the observer in the solar system. The latter consists of three components, which arise from (*one*) the revolution of the Earth around the Sun; (*two*) the rotation of the Earth on its axis; (*three*) the revolution of the Earth around the center of mass of the Earth-Moon system. This last component never exceeds ±0.014 km./sec. and may be neglected. The correction for the annual and diurnal motions of the Earth are readily computed from the formulae given by Campbell in Frost-Scheiner's *Astronomical Spectroscopy* (pp. 338–345). The values for these in the example are given respectively under v_a and v_d. Hence, the observed radial velocity of α_2 Centauri with reference to the Sun on 1911, February 27.883 (Greenwich Mean Time) was −19.55 km./sec.

Methods of reduction which depend upon dispersion formulae require an accurate knowledge of the wave-lengths of the lines used in both the comparison and stellar spectra. Accurate values of the *absolute* wave-lengths are not required but their *relative* values must be well determined. For example, a relative error of ±0.01A in the wave-length of any line would produce an error in the velocity for that line of nearly a kilometer. Interferometer measures of the wave-lengths in the spectra of a number of elements are now available, but for the wave-lengths of solar lines it is still necessary to use the determinations by Rowland. It has been shown that errors exist in Rowland's tables, amounting in some cases to as much as 0.01 or 0.02A. Another and much more serious difficulty arises, for stellar lines, from the fact that stellar spectrographs have not sufficient resolution to separate lines which were measured as separate lines by means of Rowland's more powerful instrument. It is the practice of many obser-

[3] Über die Krümmung der Spectrallinien, *Sitz. Ber. d. Math. Klasse d. k. Akad. Wien Bd.* LI, *Abth.* II, 1865; also *Frost-Scheiner, Astronomical Spectroscopy*, p. 15, 1894.

vers, where two lines merge to form one line in the star spectrum, to take the mean of the wave-lengths of the component lines, weighted according to the intensities given by Rowland for those lines in the Sun. Wave-lengths based on estimates of intensity should naturally be regarded with suspicion, and in fact we do not know, until the entire plate has been reduced, whether we have chosen an erroneous wave-length or not. It is well known that various stellar lines and blends behave differently for stars of different types. The lines in solar type stars are assumed to have the same wave-lengths as similar lines in the Sun. In the case of stars of other spectral classes, the solar lines which occur can be used in determining the wave-lengths of the non-solar lines and blends. In this manner special tables are constructed for stars of different types.

When spectrographs of lower dispersion and resolution than that of three prisms are employed for the measure of solar and later-type spectra, the effect of uncertainties in wave-length of the stellar lines, due to blends, becomes very serious. The two methods of measurement and reduction which follow eliminate the sources of error incident to the use of blends, and erroneous wave-lengths as far as it is possible to do so. The first is that due to Professor R. H. Curtiss and is called by him the 'Velocity Standard Method'. In principle it amounts to a determination of the wave-lengths of the lines in the spectrum of a source whose radial velocity is known with the particular spectrograph which is to be used for measures of stellar spectra of this same class. Thus for the measures of spectra of the solar type, a table similar to the one we have described above is formed. The micrometer readings in this table, however, are not computed from assumed wave-lengths, but are the mean of the actual settings, on comparison and solar lines, obtained on several spectrograms of the Sun. These standard plates are produced as nearly as possible under the same conditions as the stellar plates to be measured. The procedure in the reduction of the measures by means of this table is then the same as that described above. It is necessary, of course, to correct the measured stellar velocity for the radial velocity of the source when the standard spectrograms were taken. Standard tables for the reduction of measures of stars of other spectral classes may be formed in a similar manner, using as the standard sources stars whose radial velocities are well determined.

The second method is due to Professor Hartmann, and is in principle the same as the preceding one, except that the star plate is referred directly to the standard plate on a special measuring microscope, known as the spectro-comparator. The instrument is provided with two plate carriages, one of which is movable. On one of the carriages the star plate is placed and on the other, which is provided with a fine micrometer screw, is a standard plate of the Sun (taken with the stellar spectrograph). The microscope has two objectives so arranged that the images of portions of the two plates are brought, by means of total reflection prisms and a reflecting surface, to focus in the same plane and in the field of one eyepiece. By means of a silvered strip on the surface of one prism, the central portion of the Sun's spectrum is cut out and the star spectrum thrown into its place. In a similar manner, central strips of the comparison spectra of the Sun plate are replaced by those of the comparison spectra of the star plate. An ingenious arrangement of the microscopes permits of equalizing the scale of the two plates, by changing the relative magnifying powers of the two objectives. The method of measurement is, then, after proper alignment of the plates, to bring corresponding sections of the two plates into the field of the miscroscope, and by means of the micrometer screw set the corresponding lines of the comparison spectra in the same straight line. A setting is then made with the corresponding lines of the solar and star spectra in the same straight line. The difference between the micrometer readings in the two positions is the displacement of the star lines relative to the solar lines. In practice it is found sufficient to divide the length of the spectrum into about fifteen sections, for each of which these comparative settings are made. The mean of the displacements, obtained with the plates in the direct and reverse positions, when multiplied by the rVs for each section, gives for each the value $V^* - V_o$, where V^* is the radial velocity of the star and V_o that of the Sun. Theoretically, the values of $V^* - V_o$ should receive weights proportional to $\frac{1}{rVs}$ in taking the mean. Although this correction is negligible, except where an extent of spectrum of 400 or 500 A is used, its introduction leads to a very simple method of computation. Take the sum of the displacements in the direct and reverse measures and multiply by a factor $f = \frac{1}{2\Sigma \frac{1}{rVs}}$. The product is equal to the weighted mean of the values $V^* - V_o$ for each section. This, corrected for the velocity of the original Sun plate (V_o), gives the radial velocity of the star relative to the observer. The reduction to the Sun is made in the usual way. The factor f is a constant so long as the same regions are used, and its values may be computed for all combinations of the regions that are used. The great advantage of the method, aside from those which it possesses in common with the velocity standard method, is that we are able to measure and reduce in an hour a plate of a star rich in lines, and practically utilize all the material on the plate. With the older methods, to make such a comprehensive measure and reduction, *i.e.*, to utilize all of the lines on the plate, would require one or two days.

For the measures of spectra of a type other than the solar it is necessary to select for the standard plate a spectrogram of a star of that particular spectral class. In order to obtain the velocity for this standard spectrogram, it should be measured and reduced, either by the method first described or perhaps preferably by the velocity-standard method. The adopted value should be the mean of the measures made by several different observers.

The spectro-comparator offers a very efficient method in determining the differences in velocities of the same star, by measuring a series of plates of the star with reference to one of these selected as a standard.

Five of the six elements of a spectroscopic binary orbit depend only upon the accurate determination of the relative radial velocities given by the series of spectrograms. One of the most important applications of the Hartmann comparator is, therefore, to the measurement of plates of a spectroscopic binary.

For the measure and reduction of spectrograms of stars of the earlier spectral classes, the use of the Cornu-Hartmann dispersion formula will suffice, inasmuch as the spectra of such stars consist of lines due to the simple gases, the wave-lengths of which have been accurately determined in the laboratory.

The measure and reduction of spectrograms of stars of the solar and later classes of spectra are accomplished with great saving of time and labor, and by a method free from some of the uncertainties of wave-lengths, by the use of the spectro-comparator. If the observer is not provided with such an instrument the standard-velocity method is preferable to the use of the dispersion formulae, at least until a system of stellar wave-lengths of the requisite accuracy is available.

To the reader who has followed the long and intricate process of determining the radial velocity of a star, the question will naturally occur, how do we know that the final result represents the star's velocity? Obviously, the final test of the method is its ability to reproduce known velocities. Fortunately, we have at hand a means of making such a test. Since the orbital elements of the inner planets of the solar system are well determined, we can readily compute the radial velocity of one of these with reference to the Earth at any given time. It is only necessary, then, to observe the relative radial velocity of the planet and the Earth and compare this with the computed value at the time of observation. At the Lick Observatory spectrograms of *Venus* and of *Mars* are secured at frequent intervals with the stellar spectrograph and measured by the observers in the regular course of measuring stellar plates. With the three-prism spectrograph, described above, the observed and computed velocities of these two planets generally agree to within ± 0.5 km., or the unavoidable error of measure. When the spectrograms are measured by several observers, the effects of personal equation are to some extent eliminated in the mean, and an agreement within a few tenths of a kilometer is to be expected. A continual check is thus afforded on the adjustments of the spectrograph and the measurement of the spectrograms.

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CHAPTER VI

THE ORBIT OF A SPECTROSCOPIC BINARY STAR

The problem of determining the orbit of a binary system from measures of radial velocity, made in the manner described in the previous chapter, differs in several important particulars from that of computing an orbit from micrometric measures of position angle and distance. It has been shown that micrometer measures provide the data from which the projection of the orbit of the companion star with respect to its primary can be drawn, the true relative orbit following, correct in proportions but of unknown linear dimensions. The radial velocities, on the other hand, when plotted against the times, produce a periodic curve, having the general appearance of a distorted sine-curve; from this curve we are to find the elements of the true orbit of the star with respect to the center of gravity of the system of which it forms one component.[1]

Figure 7 illustrates the conditions of the problem. Let the XY-plane be tangent to the celestial sphere at the center of motion, and let the Z-axis, perpendicular to the XY-plane, be parallel to the line of sight along which the radial velocities are measured. *The velocities are considered positive* (+) *when the star is receding from, and negative* (−) *when it is approaching the observer.* The orientation of the X and Y-axes remains unknown. Let PSA be the true orbit of the star with respect to the center of motion and let the orbit plane intersect the XY-plane in the line NN'.

Then, when the star is at any point S in its orbit, its distance z from the XY-plane will be

$$z = r \sin i \sin(v + \omega)$$

[1] It is here assumed that the spectrum of only one component is visible; when both components give spectra, we may determine the relative orbit of one with respect to the other, using the same formulae but changing the value of the constant of attraction. The relative and absolute orbits are, of course, similar in every respect.

the symbols in the right hand member of the equation having the same significance as in the case of a visual binary star.

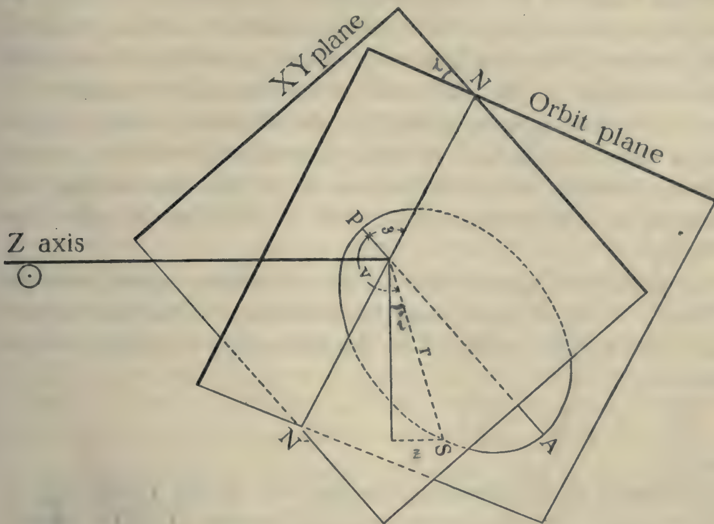

FIGURE 7

The spectrograph, however, does not give us the *distances* of the star from the XY-plane, but the *velocities of its approach to, or recession from this plane*, generally expressed in kilometers per second. The radial velocity at point S is equal to dz/dt, and is therefore expressed by

$$\frac{dz}{dt} = \sin i \sin (v + \omega) \frac{dr}{dt} + r \sin i \cos(v + \omega) \frac{dv}{dt}.$$

From the known laws of motion in an ellipse we have

$$r\frac{dv}{dt} = \frac{\mu a (1 + e \cos v)}{\sqrt{1 - e^2}}, \quad \frac{dr}{dt} = \frac{\mu a e \sin v}{\sqrt{1 - e^2}}$$

and therefore

$$\frac{dz}{dt} = \frac{\mu a \sin i}{\sqrt{1-e^2}} [e \cos \omega + \cos (v + \omega)] \qquad (1)$$

which is the fundamental equation connecting the radial velocities with the elements of the orbit.[2]

The observed velocities evidently contain the velocity, V, of the center of mass of the system, which is a constant quantity for any given simple binary system,[3] as well as the variable velocities due to the star's orbital motion and the quantity V must therefore be subtracted from the observed values to make them purely periodic. In other words, the velocity curve is purely periodic only with respect to a line representing the velocity of the system as a whole. This line is called the V-axis.

Equation (1) applies only to the velocities counted from the V-axis. If $d\zeta/dt$ represents the velocity as actually observed (*i.e.*, the velocity referred to the zero-axis) we shall have the relation [4]

$$\frac{d\zeta}{dt} = V + \frac{dz}{dt}. \quad \text{(1a)}$$

Methods of determining the position of the V-axis will be given later; for the present we shall assume it to be known.

Five constants enter the right hand member of equation (1), *viz.*, $a \sin i$, e, μ, ω and (through v) T. These express the five orbit elements which it is possible to determine by measures of radial velocity.

[2] In place of $(v + \omega)$ the symbol u (= the argument of the latitude) is often used, the expressions for $\frac{dr}{dt}$ and $r\frac{du}{dt}$ written

$$\frac{dr}{dt} = \frac{f}{\sqrt{p}} e \sin (u - \omega), \text{ and } r\frac{du}{dt} = \frac{f\sqrt{p}}{r} = \frac{f}{\sqrt{p}} [1 + e \cos (u - \omega)]$$

and hence the fundamental equation in the form

$$\frac{dz}{dt} = \frac{f}{\sqrt{p}} \sin i (\cos u + e \cos \omega).$$

In these equations $p [= a(1 - e)]$ is the semi-parameter of the true ellipse and f denotes the constant of attraction, which, when the spectrum of only one component is visible, and the motion is determined with reference to the center of mass of the system, takes the form $\frac{k\, m^{3/2}}{m + m_1}$, k being the Gaussian constant; when both spectra are visible and the motion of one star with respect to the other is determined, $f = k\sqrt{m + m_1}$. It is clear that the form of the fundamental equation will be the same whatever value we may assign to f and the constant of attraction may therefore be disregarded until the question of the relative masses in the system comes up for discussion.

[3] In a triple or multiple system, this quantity will itself be variable.

[4] The symbol γ is often used for the velocity of the system instead of V.

Since the inclination of the orbit plane is not determinable, the value of a, the semi-major axis, must also remain unknown. It is therefore customary to regard the function $a \sin i$ as an element. Further, it is clear that the position of the line of nodes cannot be determined though we can find the times when the star passes through each of the nodal points. The various elements have the same definitions as in the case of visual binary star orbits (see page 72) except that the angle ω in spectroscopic binary orbits is always measured from the ascending node, *the node at which the star is moving away from the observer.* It will be seen later that the radial velocity has its maximum positive value at this node and its minimum positive value (or maximum negative value) at the descending node. It should also be noted that the unit of time for μ (and therefore for P) is the day, not the year as in visual binary orbits.

Theoretically, values of the radial velocity at five different times suffice for the complete solution of equation (1); practically, no computer undertakes an orbit until a considerable number of measures is available which give the velocities at short intervals throughout the entire revolution period. To secure a satisfactory distribution of the observations a preliminary value of the period is necessary and such a value can ordinarily be obtained without difficulty by plotting the early observations on coordinate paper, taking the times, expressed in Julian days and decimals of a day, as abscissae and the velocities, expressed in kilometers per second, as ordinates. A convenient epoch as origin for the period is selected near the beginning of the series, preferably one corresponding to a point of maximum or minimum velocity. If later measures indicate that the period is in error, a new period which is a submultiple of the original one will often prove satisfactory. In difficult cases, the following artifice may be found helpful.[5] Copy from one-third to one-half of the series of observed points, choosing the time interval best covered by observa-

[5] This was suggested to me by Dr. R. K. Young who says that it has been used with good results by several computers of binary star orbits at the Dominion Observatory. No mention of the device has been found in print and its author is unknown to me. Its usefulness arises from the fact that, in effect, it doubles the number of observations for a given time interval.

tion, on transparent paper; slide the copy along the original plot, keeping the time-axis in coincidence, until some point on the copy falls approximately upon a different point in the original at which the velocity is changing in the same direction. The time interval between the two points is evidently equal to the period or a multiple of the period.

Schlesinger [6] has published a criterion that may be applied to advantage in cases where an observer has accumulated many plates of a star which apparently shows variable radial velocity without being able to determine any period. It consists in constructing a frequency curve for the velocities by "dividing the total range exhibited by the measured velocities into successive groups of equal extent, say three kilometers each, and then counting the number of velocities that fall within these groups. Regarding these numbers as ordinates, we plot them and join the ends by a smooth curve." This curve is compared with the well-known error-curve; if the two are the same, within reasonable limits, we may conclude that the differences in the measured velocities are due to errors of observation, and afford no support for the assumption that the star is a spectroscopic binary. If the two curves differ, the star is a binary and the form of the frequency curve will give an idea as to the general character of the orbit and frequently furnish a clew to the period. For Schlesinger shows that circular orbits, elliptic orbits with periastron at descending node, elliptic orbits with periastron at ascending node, and elliptic orbits with periastron removed 90° from the nodes, all have characteristic frequency curves which differ in form from the error-curve. When the nature of the frequency curve has shown to which of these classes the orbit in question belongs, it becomes very much easier to decide upon the epochs for the various observed velocities, and thus upon an approximate value for the period.

When the period is approximately known all of the observations may be reduced to a single revolution by subtracting

[6] *Astrophysical Journal*, vol. *41*, p. 162, 1915. In his paper on the "Orbit of the Spectroscopic Binary χ *Aurigae*" (*Journal R. A. S. C.* vol. X, p. 358), Young shows that the errors of measurement affect the expected distribution in such a manner as to mask to a considerable degree the presence of the orbital variation.

multiples of the period from the later dates. A preliminary curve is drawn to represent the plotted positions as closely as possible. The deviations from the curve at points near the mean of the maximum and minimum velocities, where a change in the periodic time will have the greatest effect, will indicate advisable changes in the assumed period and these are readily found by dividing the deviations of such critical observations, expressed in time, by the number of revolutions elapsed. A second curve is then drawn whose periodic time will generally be very close to the true value. In practice it will frequently happen that two or three measures of the radial velocity of a star are available which were made (perhaps at another observatory) several years before the series of spectrograms for the orbit computation is begun. When an approximate value of the period has been found from the later series, these early plates will determine its true value with high precision. Generally they are not used in finding the other orbit elements.

When the period has been determined as accurately as possible and a series of spectrograms has been accumulated giving the velocities at points well distributed throughout the entire period, the most probable curve is drawn, by estimation, through the points as plotted, and, if the ingenious methods of superposition devised by Schwarzschild and Zurhellen are to be used, the curve should be prolonged through a revolution and a half. The plotted points used for this curve should represent normal positions, formed by combining several velocities observed at very nearly the same orbit phase, whenever the number of observations is sufficient to permit such combinations. In making the combinations, the question of weights arises, and here the practice of computers varies considerably, for several factors enter. The character of the lines on the spectrograms, broad or narrow, sharp or ill-defined, strong or weak, is one factor; the number of lines is another; if the plates have been taken with different telescopes and spectrographs, a third factor is introduced. These must all be considered in assigning the weights to each plate. The only direction that can be given is the general one to use rather a simple system of weighting. It will rarely be of advantage to assign fractional weights, or to use a range of weights greater than, say, four units. The weights should, of course, be assigned to each plate, at the time of measurement.

The errors in drawing the most probable curve have considerable effect upon the accuracy of the determination of the elements. At best the curve is not likely to be a perfect representation of the elliptic motion which caused it since it is natural to bend the curve slightly in or out at different points to satisfy the more or less exact observations. This difficulty is inherent and for it there is apparently no remedy other than that of testing the first orbit by a trial ephemeris and making the small changes in the elements which are indicated by the residuals.[7]

If Figure 8 represents a velocity curve, it is evident from equation (1) that the points A and B correspond respectively to the ascending and descending nodes of the star's orbit, for at the times of nodal passage we have $(v + \omega) = 0°$, and $(v + \omega) = 180°$, respectively and therefore $\cos(v + \omega) = \pm 1$. The radial velocity thus reaches its maximum and minimum values at the nodal points.

Taking A and B as the magnitudes of the curve-ordinates at the points of maximum and minimum reckoned from the V-axis, regarding B as a positive quantity and writing for brevity $K = \dfrac{\mu\, a \sin i}{\sqrt{1 - e^2}}$, we have

$$A = K(1 + e\cos\omega)$$
$$B = K(1 - e\cos\omega)$$

and therefore

$$\left.\begin{aligned} \frac{A+B}{2} &= K \\ \frac{A-B}{2} &= K\,e\cos\omega \\ \frac{A-B}{A+B} &= e\cos\omega \end{aligned}\right\} \qquad (2)$$

[7] King's method affords a graphical test of the first orbit found, see page 154.

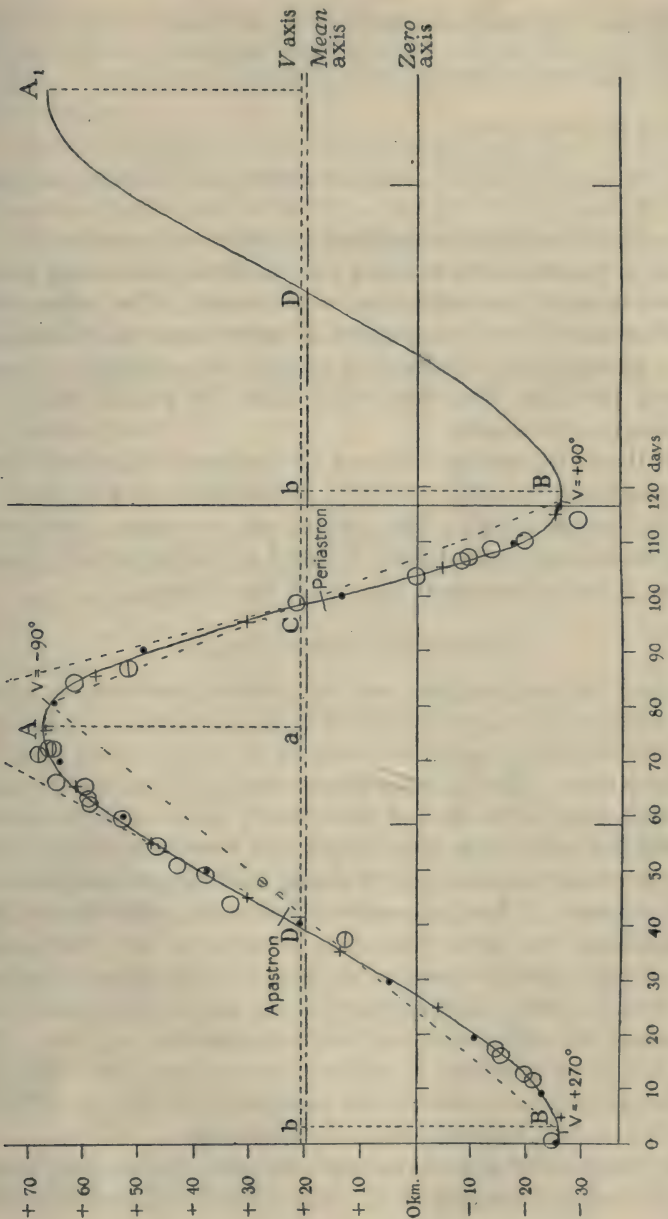

FIGURE 8. Velocity Curve of κ Velorum

Hence we may write equation (1) in the form

$$\frac{dz}{dt} = K[e\cos\omega + \cos(v+\omega)] = \frac{A-B}{2} + \frac{A+B}{2}\cos(v+\omega) \qquad (3)$$

and (1a) in the form

$$\frac{d\zeta}{dt} = V + \frac{A-B}{2} + \frac{A+B}{2}\cos(v+\omega) = V_1 + \frac{A+B}{2}\cos(v+\omega). \quad (3a)$$

K is therefore the half-amplitude of the velocity curve.

Up to the point now reached practically all methods of spectroscopic orbit determination are identical. But when the fundamental relations are given as above, and the curve has been drawn, quite a variety of methods is available for computing the orbit elements, other than the period, which is assumed to be known.

Of these, the method devised by Lehmann-Filhés will first be presented, essentially in full; other methods will then be treated in less detail. The student who desires to study the various methods more fully is referred to the important papers given in the references at the end of the chapter.

METHOD OF LEHMANN-FILHÉS

Given the observations, and the velocity curve drawn with the value of P assumed as known, the first step is to fix the V-axis, the line defining the velocity of the center of gravity of the system. This is found by the condition that the integral of dz/dt, that is, the area of the velocity curve, must be equal for the portions of the curve above and below the V-axis. By far the easiest method of performing this integration is to use a planimeter. A line, approximately correct, is first drawn by estimation; the areas contained between it and the curve above and below are measured, and the difference between the two is taken. The position of the axis is then shifted to eliminate this difference, and the measures are repeated. It will rarely be necessary to make more than one correction to secure an accurate value of the position of the V-axis, which, by this method, depends upon the entire curve.

If a planimeter is not available, the areas above and below the axis may be equalized by using coordinate paper for the plot of the curve and counting the small squares in each area. Approximate mechanical integration, as advised by Lehmann-Filhés, may also be resorted to, by those who enjoy this form of recreation.

Having found the V-axis, the ordinates to it are next drawn from the points of maximum and minimum velocity, A and B. It is at this point, as Curtis says, that the method is weakest, for slight errors in fixing the position of A and B may easily arise. It is well to apply the check afforded by the requirement that area AaC (Figure 8) must equal CbB and DaA equal BbD. Since C and D lie on the V-axis the velocities at these points are zero, hence from equations (3) and (2) we have for $\frac{dz}{dt}$ at these points

$$\cos (v + \omega) = -\frac{A-B}{A+B} = -e \cos \omega \qquad (4)$$

If v_1 is the true anomaly corresponding to the point C, which is traversed by the star on the way from the ascending to the descending node, and v_2, the true anomaly for the point D, $\sin (v_1 + \omega)$ will be positive, $\sin (v_2 + \omega)$ negative, and we shall have

$$\left.\begin{aligned} \cos (v_1 + \omega) &= -\frac{A-B}{A+B}, \ \cos (v_2 + \omega) = -\frac{A-B}{A+B} \\ \sin (v_1 + \omega) &= \frac{2\sqrt{AB}}{A+B}, \ \sin (v_2 + \omega) = -\frac{2\sqrt{AB}}{A+B} \end{aligned}\right\} \qquad (5)$$

Let Z_1 and Z_2 denote the areas [8] AaC and bBD (Figure 8) respectively, and let r_1 and r_2 be the radii vectores for the points C and D.

Then

$$Z_1 = r_1 \sin i \sin (v_1 + \omega)$$
$$Z_2 = r_2 \sin i \sin (v_2 + \omega) = -r_2 \sin i \sin (v_1 + \omega)$$

and therefore

$$-\frac{Z_1}{Z_2} = \frac{r_1}{r_2} = \frac{1 + e \cos v_2}{1 + e \cos v_1} \qquad (6)$$

[8] These areas represent the distances of the star from the XY-plane at the points in its orbit corresponding to $(v_1 + \omega)$ and $(v_2 + \omega)$.

since $r = \dfrac{a(1-e^2)}{1+e\cos v}$. Write $(v + \omega - \omega)$ for v, in (6), expand, and reduce, with the aid of the relations in (5) and (4), and we have

$$-\frac{Z_1}{Z_2} = \frac{\sin(v_1+\omega) - e\sin\omega}{\sin(v_1+\omega)+e\sin\omega},$$

whence

$$e\sin\omega = \frac{Z_2+Z_1}{Z_2-Z_1}\sin(v_1+\omega) = \frac{2\sqrt{AB}}{A+B}\cdot\frac{Z_2+Z_1}{Z_2-Z_1}. \qquad (7)$$

Equations (7) and the last of (2) determine e and ω. The values of A and B are taken from the curve, and the areas Z_1 and Z_2 are quickly integrated from the curve portions AaC and bBD by means of a planimeter, the latter area being regarded as negative in sign. Since the areas enter as a ratio, the unit of area used is entirely immaterial.

At the time of periastron passage $v = 0°$; hence from equation (3) we have

$$\frac{dz}{dt_p} = K(1+e)\cos\omega \qquad (8)$$

which gives the ordinate corresponding to the point of periastron passage. Two points of the curve will have the same ordinate, but since $(v + \omega)$ equals 0°, 180°, and 360° for the points A, B, and A_1, respectively, there will be no ambiguity as to the position of the periastron point. The abscissa of this point, properly combined with the epoch chosen for the beginning of the curve, defines T, the time of periastron passage. Instead of using (8) we may find T by determining E for the point C for which the value of v is known, and then employ the formulae

$$\left.\begin{aligned}\tan \tfrac{1}{2} E &= \sqrt{\frac{1-e}{1+e}}\tan\tfrac{1}{2}v\\ T &= t - \frac{E - e\sin E}{\mu}\end{aligned}\right\} \qquad (8a)$$

or, if the eccentricity is less than 0.77, M may be taken directly from the Allegheny Tables, and T found from the relation

$$M = \mu(t - T). \qquad (8b)$$

Such procedure is especially advisable when the periastron point falls near points A or B on the curve.

By definition (page 140) we have

$$K = \frac{\mu\, a \sin i}{\sqrt{1 - e^2}}$$

and hence

$$\mu = \frac{K\sqrt{1 - e^2}}{a \sin i} = \frac{A + B}{2} \cdot \frac{\sqrt{1 - e^2}}{a \sin i}$$

from which we may find the value of the product $a \sin i$. Since the unit of time for A and B is the second, while for μ it is the day, the factor 86,400 must be introduced. Our equation then becomes

$$a \sin i = 86{,}400 \frac{K}{\mu} \sqrt{1 - e^2} = [4.13833] KP \sqrt{1 - e^2} \qquad (9)$$

the number in brackets being the logarithm of the quotient $86400 \div 2\pi$.

Summarizing, the practical procedure is:

1. Find the period as accurately as possible by successive trials and plot the most probable velocity curve on the basis of normal places.
2. Find the position of the V-axis by integration of areas, using the planimeter, if available.
3. Measure the ordinates for points A and B and find the areas of AaC and bBD expressed in any convenient units.
4. From (2) and (7) determine K, e, and ω.
5. From (8), or by calculation from the value of v, for the point C, determine T.
6. From (9) determine $a \sin i$.

To test the elements by comparison with the observations, we compute the radial velocity for each date by the formulae:

$$\left.\begin{aligned} M &= \mu(t - T) = E - e \sin E \\ \tan \tfrac{1}{2} v &= \sqrt{\frac{1 + e}{1 - e}} \tan \tfrac{1}{2} E \\ \frac{d\zeta}{dt} &= V + Ke \cos \omega + K \cos (v + \omega) \end{aligned}\right\} \qquad (10)$$

The value of v for each value of M may be taken directly from the Allegheny Tables, if e is less than 0.77.

To illustrate Lehmann-Filhés's method I have chosen the orbit computed for *κ Velorum*, by H. D. Curtis, the velocity curve for which is given in Figure 8.

The observations used were as follows:

Julian Day, G. M. T.	Vel.	Julian Day, G. M. T.	Vel.
2416546.739	+ 68.5km	2417686.591	+ 33.8km
60.703	+ 12.9	91.572	+ 38.2
97.651	+ 65.7	92.545	+ 43.2
6912.601	+ 53.3	96.480	+ 46.7
7587.844	+ 58.6	7701.494	+ 52.7
88.788	+ 57.9	41.466	+ 22.1
90.829	+ 58.5	46.463	+ 0.3
91.824	+ 64.8	49.470	− 7.6
97.788	+ 65.8	50.479	− 8.8
7609.790	+ 62.0	51.463	− 13.3
54.534	− 21.0	53.457	− 19.2
55.556	− 19.2	58.451	− 29.0
58.570	− 15.2	59.460	− 24.6
59.545	− 14.5		

The small circles representing the first four observations, which are important in determining the period, owing to their distance in time from the later ones, are barred in the diagram. The period, P, was assumed to be 116.65 days, and the beginning of the curve is at Julian day 2416476.0. This is not exactly at a minimum, as may be seen from the diagram.

From measures of the curve we find

$A = 46.3$ $Z_1 = AaC = +0.168$

$B = 46.9$ $Z_2 = bBD = -0.259$

$A + B = 93.2$ $Z_2 + Z_1 = -0.091$

$A - B = -0.6$ $Z_2 - Z_1 = -0.427$

$K = (A + B)/2 = 46.6$

The solution of equations (2), (7), (8) then proceeds as follows:

log 2	0.3010	log $(1 + e)$	0.0828
log $\sqrt{AB}$	1.6684	log cos ω	8.4800n
colog $(A + B)$	8.0306	log K	1.6684
log $(Z_2 + Z_1)$	8.9560n	log $\left(\frac{dz}{dt_p}\right)$	0.2312n
colog $(Z_2 - Z_1)$	0.3696n		
log e sin ω	9.3286	ordinate p	− 1.7 km
log $(A - B)$	9.7782n	∴ from curve t_p = 98.4	
log $\left(\frac{A-B}{A+B}=\right)$ $e \cos\omega$	7.8088n	T[9] = J. D. 2416457.75	
log tan ω	1.5198n	log const.	4.1383
ω	91.73°	log K	1.6684
sin ω	9.9998	log P	2.0669
		log $\sqrt{1 - e^2}$	9.9902
log e	9.3288	log a sin i	7.8638
e	0.21	a sin i	73,000,000 + km

The preliminary values thus obtained are next tested by comparing the velocities derived from them by equations (10) with the observed velocities. To illustrate, let us compute the velocity for J. D. 2416496.0, twenty days after the origin adopted in our curve. We have

t = 2416496.0		log cos $(v + \omega)$	9.8277n
$t - T$ =	+ 38.0	log K	1.6684
log $(t - T)$	1.57978		1.4961n
log μ	0.48942	$\frac{A+B}{2}\cos(v+\omega)$	− 31.3 km
M	117.27°	$\frac{A-B}{2}$	− 0.3
v	136.01	V	+ 20.7
$v + \omega$	227.74	$\frac{d\zeta}{dt}$	= − 10.9 km.

[9] T is here taken one revolution earlier than the date for the periastron point marked on the curve. Using equation (8a), or (8b) we obtain T = J. D. 2416458.0 which is adopted.

In this manner we compute as many velocities as necessary to obtain a curve for comparison with the observed velocity curve. In the present instance this was done for every tenth day, and the results plotted as heavy black dots in the figure. By noting the discrepancies, it appears that the branch on the apastron side of the computed curve, if drawn, would be a little too sloping, the other branch too steep, which indicates that the computed value of e is a little too large. Changing this element and making the corresponding slight changes required in T and ω, the test was repeated, and after a few trials, the following elements were adopted as best representing the observations:

$$\begin{aligned} V &= +21.9\text{km} \\ P &= 116.65 \text{ days} \\ e &= 0.19 \\ K &= 46.5 \\ \omega &= 96.23^\circ \\ T &= \text{J. D. } 2416459.0 \\ a \sin i &= 73{,}000{,}000\text{km} \end{aligned}$$

The correction to the value of V was found last of all from the residuals of the final ephemeris by the simple formula $\frac{[v]}{n}$, where n is the number of observations and v the residual, o − c. The residuals from the final ephemeris and the final curve may be found in *Lick Observatory Bulletin*, No. 122, 1907.

Lehmann-Filhés's method may be termed the classical one, and it is probably more generally used than any other. The method proposed by Rambaut is considerably longer and more involved than the later ones, and for that reason will not be described here. Wilsing's method, as originally published, was suitable only for orbits of small eccentricity, but Russell later extended it to make it applicable to larger eccentricities as well. This method is purely analytical, consisting in finding a Fourier's series for the velocity in terms of the elements. It should be very useful in special cases, particularly when the period is so nearly a year that one part of the velocity curve is not represented by any observations; but it is considerably longer, in time consumed, than the method of Lehmann-Filhés and other geometrical methods to be described presently, and it will not be further considered here.

Certain features of the methods proposed by Schwarzschild and Zurhellen are both ingenious and practical. The following account of them is taken in substance from Curtis's article already referred to.

SCHWARZSCHILD'S METHOD

Given the velocity curve and the period, Schwarzschild first determines the time of periastron passage. Let M_1 and M_2 be the observed velocities (*i.e., the velocities measured from the zero-axis*) of maximum and minimum, and draw the line whose ordinate is $\frac{M_1 + M_2}{2}$. This line is the mean axis. Mark upon it the points corresponding to $P/2$ and $3P/2$; then lay a piece of semi-transparent paper over the plot, copy upon it the curve together with the mean axis and mark also the points 0, $P/2$, P, and $3P/2$. Shift the copy bodily along the mean axis for the distance $P/2$, and then rotate it 180° about this axis,—*i.e.*, turn the copy face downward on the original curve keeping the mean axis in coincidence and bring the point 0 or P of the copy over the point $P/2$ of the original. The curves will then cut each other in at least four points, and, in general, in four points only. These will fall into two pairs, the points of each pair separated by an abscissa interval $P/2$. The points of one pair will be on different branches of the velocity curve, and it is easy to see that, if v_1 and v_2 represent their true anomalies, we shall have $v_2 = v_1 + 180°$. Now the only two points in the true orbit which are separated by one-half a revolution and for which at the same time this relation of the true anomalies holds are the points of periastron and apastron passage. Hence, to select these points, choose the two points of intersection of the curve and its copy which are separated by half a revolution and which lie on different branches of the curve. To distinguish periastron from apastron we have the criteria: *First*, at periastron the velocity curve is steeper with respect to the axis than at apastron; *Second*, the curve is for a shorter time on that side of the mean axis on which the point of periastron lies.

This method is exceedingly good except when the eccentricity is small. In this case ω and T are quite indeterminate and small errors in drawing the velocity-curve will be very troublesome. The method of Lehmann-Filhés is then to be preferred.

Having the value of T, the value of ω is next found as follows: From equations (1a) and (3) it is readily seen that the position of the mean axis is

$$\frac{M_1 + M_2}{2} = V + K\,e \cos \omega = V_1 \,,$$

and that, accordingly, the ordinate z' of any point measured from the mean axis is

$$z' = \frac{d\zeta}{dt} - V_1 = K \cos (v + \omega). \qquad (11)$$

Now at periastron $v = 0°$, at apastron $v = 180°$. Hence, if we call the ordinates from the mean axis for these points z'_p and z'_a we shall have

$$\cos \omega = \frac{z'_p}{K} \text{ or } \cos \omega = \frac{z'_p - z'_a}{2K} \qquad (12)$$

from which to determine ω. This method is at its best when ω is near 90°.

Zurhellen has simplified Schwarzschild's method of finding e, and we shall give this simpler form in connection with Zurhellen's simple method of finding ω.

ZURHELLEN'S METHODS

Zurhellen's method of determining ω depends upon the relations between the velocities for the two orbit points whose true anomalies are $\pm 90°$. From equation (11) we have, when $v = -90°$,

$$z_1 = + K \sin \omega$$

and when $v = + 90°$,

$$z_2 = - K \sin \omega.$$

Moreover, for these two points we have

$$E_1 = -E_2$$
$$M_1 = -M_2$$
$$(t_1 - T) = -(t_2 - T);$$

hence the two points are symmetrically placed with respect to the mean axis in the Y-coordinate and with respect to the point of periastron passage in the X-coordinate. They may therefore be determined by rotating the curve copy through 180° about the intersection of the ordinate of periastron with the mean axis, and noting the two points of intersection of the copy with the original curve. If the curve is prolonged through one and one-half revolutions, another point 180° from one of these, say at $+ 270°$, can be determined in similar manner and the location of all three can then be checked by drawing the lines connecting the point $v = +270°$ with $v = -90°$, and $v = -90°$ with $v = +90°$. These lines should cut the mean axis at its intersections with the ordinates of periastron and apastron respectively. From the ordinates of the two points $v = \pm 90°$, measured from the mean axis, we have

$$\sin \omega = \frac{z_1 - z_2}{2K}, \text{ or } \tan \omega = \frac{z_1 - z_2}{z_p - z_a} \qquad (12)$$

from which to find ω. The method is at its best when ω is small.

Zurhellen's simplification of Schwarzschild's method of finding e is also based upon the relations between the two points $v = \pm 90°$. Since

$$\tan \tfrac{1}{2} E = \tan \tfrac{1}{2} v \tan (45° - \tfrac{1}{2} \phi)$$

where ϕ is the eccentric angle, we have, when $v = \pm 90°$,

$$E_1 = -(90° - \phi), \quad E_2 = +(90° - \phi).$$

Similarly,

$$M_1 = -(90° - \phi) + \frac{\sin \phi \sin (90° - \phi)}{\sin 1''}$$

$$M_2 = +(90° - \phi) - \frac{\sin \phi \sin (90° - \phi)}{\sin 1''}$$

and therefore

$$M_2 - M_1 = \frac{360°}{P}(t_2 - t_1) = (180° - 2\phi) - \frac{\sin (180° - 2\phi)}{\sin 1''}. \quad (13)$$

The value of $(t_2 - t_1)$ may be read off directly from the diagram, and the value of $(90° - \phi)$ can then be taken from the table for equation (13), computed by Schwarzschild, which is given below. Like the above method for finding ω this method is best when ω is small.

Schwarzschild's Table for the Equation

$$2\,\eta - \sin 2\,\eta = \frac{360°}{P}(t_2 - t_1).$$

η	$\frac{t_2-t_1}{P}$	η	$\frac{t_2-t_1}{P}$	η	$\frac{t_2-t_1}{P}$
0°	0.0000	30°	0.0290	60°	0.1956
1	0.0000	31	0.0318	61	0.2040
2	0.0000	32	0.0348	62	0.2125
3	0.0000	33	0.0380	63	0.2213
4	0.0001	34	0.0414	64	0.2303
5	0.0001	35	0.0450	65	0.2393
6	0.0002	36	0.0488	66	0.2485
7	0.0004	37	0.0527	67	0.2578
8	0.0006	38	0.0568	68	0.2673
9	0.0008	39	0.0611	69	0.2769
10	0.0011	40	0.0656	70	0.2867
11	0.0015	41	0.0703	71	0.2966
12	0.0020	42	0.0751	72	0.3065
13	0.0025	43	0.0802	73	0.3166
14	0.0031	44	0.0855	74	0.3268
15	0.0038	45	0.0910	75	0.3371
16	0.0046	46	0.0967	76	0.3475
17	0.0055	47	0.1025	77	0.3581
18	0.0065	48	0.1085	78	0.3687
19	0.0077	49	0.1147	79	0.3793
20	0.0089	50	0.1212	80	0.3900
21	0.0103	51	0.1278	81	0.4008
22	0.0117	52	0.1346	82	0.4117
23	0.0133	53	0.1416	83	0.4226
24	0.0151	54	0.1488	84	0.4335
25	0.0170	55	0.1561	85	0.4446
26	0.0191	56	0.1636	86	0.4557
27	0.0213	57	0.1713	87	0.4667
28	0.0237	58	0.1792	88	0.4778
29	0.0262	59	0.1873	89	0.4889
30	0.0290	60	0.1956	90	0.5000

Zurhellen also gives a method for finding the eccentricity by drawing the tangents to the curve at the points of periastron. These can be drawn quite accurately except when the periastron falls near a maximum or a minimum of the curve. Slight changes in its position will then introduce considerable changes in the inclinations of the tangent lines.

The expression for the slope of a tangent may be written

$$\frac{dx}{dt} = \frac{2\pi}{P} \cdot \frac{dx}{dM} = \frac{2\pi}{P} \cdot \frac{1}{1 - e\cos E} \cdot \frac{dx}{dE}$$

where $x\left(= \frac{dz}{dt}\right)$ represents the ordinate drawn to the V-axis.

Also, by introducing the known values

$$\cos v = \frac{\cos E - \sin\phi}{1 - e\cos E}, \quad \sin v = \frac{\cos\phi \sin E}{1 - e\cos E}$$

and transforming and simplifying we may write the fundamental equation (3) in the form

$$x = \frac{dz}{dt} = K\cos\phi \cdot \frac{\cos\phi\cos\omega\cos E - \sin\omega\sin E}{1 - e\cos E}.$$

Differentiating with respect to E, substituting and reducing, we have

$$\frac{dx}{dt} = \frac{2\pi}{P}K\cos\phi \cdot \frac{-\cos\phi\cos\omega\sin E - \sin\omega\cos E + e\sin\omega}{(1 - e\cos E)^3}. \quad (14)$$

At periastron $E = 0°$ and at apastron $E = 180°$, whence we have

$$\frac{dx}{dt_p} = \frac{-2\pi K\cos\phi\sin\omega}{P(1 - e)^2}, \quad \frac{dx}{dta} = \frac{+2\pi K\cos\phi\sin\omega}{P(1 + e)^2},$$

and therefore

$$\frac{dx}{dt_p} \Bigg/ \frac{dx}{dt_a} = -\frac{(1 + e)^2}{(1 - e)^2} = -q^2,$$

whence

$$e = \frac{q - 1}{q + 1}. \quad (15)$$

KING'S METHOD

The methods of orbit computation so far described in this chapter all rest upon the curve drawn to represent as closely as possible the observed velocities and, at the same time, to satisfy the conditions for elliptic motion. Unless the measures are very precise, the first approximation will ordinarily not be satisfactory. As stated on page 140, the only remedy is to compute an ephemeris from the elements and, on the basis of the residuals thus found, to draw a new curve. This process is sometimes repeated three or four times before a curve is found which will yield elements upon which a least squares solution may be based.

The method devised by Dr. King, which is now to be presented, aims to substitute a rapid graphical process for testing the preliminary curve. Dr. King shows that a circle having its center on the mean axis and a radius equal to K, the semi-amplitude of the velocity curve, "may be used as the equivalent of the hodograph of observed velocities."[10]

Let the velocity curve and the circle be drawn (see Figure 9) and the abscissa distance corresponding to one revolution (P being assumed to be known) be divided into any convenient number of parts, say forty.[11] Now mark consecutive points on the circumference of the circle by drawing lines parallel to the mean axis at the intersections of the velocity curve with the ordinates corresponding to successive values of the abscissa and extending them to the circle. The circumference will be divided into forty unequal parts, but these inequalities will be found to vary uniformly. "The points will be close together in the vicinity of one point of the circle, and will gradually separate as we proceed in either direction therefrom, until at the diametrically opposite point they reach their maximum distance apart." These unequal arcs of the circle correspond to the increase in the true anomalies in the orbit in the equal time intervals, and therefore the point of widest

[10] For the proof of this relation the reader is referred to the original article, *Astrophysical Journal*, vol. 27, p. 125, 1908.

[11] An even number should be chosen, and it is obviously most convenient to make the drawing upon coordinate paper.

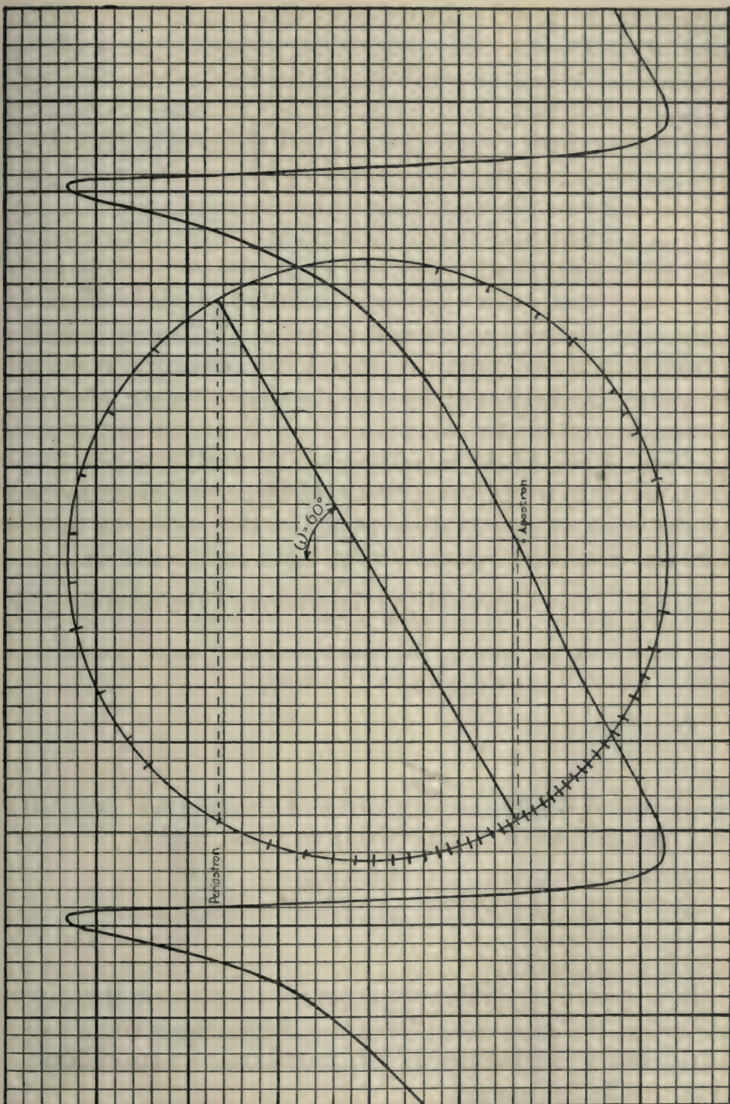

FIGURE 9. King's Orbit Method. Graph for $e = 0.75$, $\omega = 60°$

separation of the circle divisions corresponds to periastron, that of least separation, to apastron. Further, the angle between the *Y*-axis and the periastron-half of the diameter between these two points is equal to ω. To locate the point of periastron on the velocity curve, find the intersection of the steeper branch of the curve with a line drawn from the periastron point on the circle parallel to the mean axis.

It is evident that the division points of the circumference will be symmetrically disposed with respect to the apsidal diameter (the diameter joining periastron and apastron points) only when one of the division points in the line of abscissae corresponds to an apse. In general, the periastron point will lie within the longest division of the circumference, the apastron point within the shortest. If desired, the approximate position of one of these points may be used as a new origin from which to set off the fortieths of the period along the axis of abscissae, and two division points on the circle may then be brought into closer coincidence with the apsidal points.

Since $\frac{dv}{dt}$ varies inversely as the square of the distance from the focus, by measuring the lengths d_1 and d_2 of the arcs at points where v equals v_1 and v_2, we have

$$\frac{d_1}{d_2} = \frac{(1 + e \cos v_1)^2}{(1 + e \cos v_2)^2},$$

and hence if the arcs are measured at the points of periastron and apastron where v respectively equals 0° and 180°,

$$\frac{d_1}{d_2} = \left(\frac{1 + e}{1 - e}\right)^2, \text{ or } e = \frac{\sqrt{d_1} - \sqrt{d_2}}{\sqrt{d_1} + \sqrt{d_2}} \qquad (16)$$

which determines e.

It is generally sufficiently accurate to measure the chords instead of the arcs; when the eccentricity is high and the arcs at periastron inconveniently long, additional points of division may readily be inserted.

It will be observed that this process furnishes a more thorough test of the accuracy of the graph (velocity curve) than the method of equality of areas. If it is imperfect, the points on the circumference of the circle will not be distributed according to the regular order of increase or decrease of the included arcs. If an ordinate of the graph is too long or too short, the corresponding point on the circumference will be too near to or too far from the vertical diameter. If the points of maximum and minimum velocity have not been well determined, the diameter of the circle will be too long or too short. In the former case all the points of the circumference will be crowded away from the vertical diameter; in the latter, toward it.[12]

To test a given set of elements by comparison with the observations proceed as follows:

Construct a circular protractor on some semi-transparent material (*e.g.*, celluloid or linen tracing cloth) and divide it into forty parts by radii to points on the circumference representing the true anomalies for the given value of e corresponding to every 9° of mean anomaly (*i.e.*, to fortieths of the period). If the eccentricity is less than 0.77 the values of the true anomaly can be taken directly from the Allegheny Tables.

On the plot of the orbit draw a circle of radius K with its center on the mean axis and draw its vertical diameter. Set the protractor upon the circle, making the centers coincide, and turn the apsidal diameter of the protractor until it makes an angle equal to ω with the vertical diameter. Now note the points where the radial lines representing the anomalies intersect the circumference of the circle. The abscissa axis of the plot also having been divided into forty equal parts, erect perpendiculars at the points of division equal to the corresponding ordinates of the circle. A freehand curve through the extremities of these perpendiculars (*i.e.*, ordinates to the mean axis) gives the computed curve or 'ephemeris', and the residuals can be read directly from the plot. The advantage of using coordinate paper will be obvious.

From the account just given it will appear that King's method is longer, or at least not shorter, than the others described if only a single orbit is to be computed. But when orbit computation is to be taken up as a part of a regular program of work, the method has very decided advantages. It is then to be used as follows:

[12] King, *loc. cit.*

Let a set of protractors be constructed on transparent celluloid with radii representing the divisions in true anomaly for every 9° of mean anomaly for the values $e = 0.00$, $e = 0.05$ to $e = 0.95$.

With the aid of these protractors draw curves on tracing linen representing orbits with all values of e from 0.00 to 0.95 and all values of ω from 0° to 360°. The intervals for e should be 0.05, save for the larger values which are seldom used, and for ω, 15°. Practically, values of ω to 90° will suffice, the curves for the values in the remaining quadrants being obtained by inverting the sheet, and by looking through the linen from the back in the two positions. Given the protractors, a complete set of curves may be constructed in about ten hours' time.

Having such a set of curves, plot the normal places for any given binary star *on the same scale as these curves in time and in velocity.*[13] Now place the standard curves upon the plot until one is found that fits the observations. "If two or more curves seem to give about equally good representations, it is quite possible to interpolate elements between the graphs plotted."[14]

By this process values of e correct to within one or two hundredths and of ω correct within a few degrees can generally be obtained at the first trial and with an expenditure of less than ten minutes' time. The time of periastron passage follows at once, and this set of preliminary elements may then be used as the basis for a least squares adjustment. The procedure has been found very satisfactory at the Dominion Observatory at Ottawa.

RUSSELL'S SHORT METHOD

Professor Henry Norris Russell has devised a graphical method which is equally simple in its practical application.[15]

[13] Since the velocity curve is ordinarily based on from fifteen to twenty normal places the work of multiplying by the appropriate reduction factors will require a very few minutes only. Of course the amplitude of the curve as well as the period must be known before the reduction factors can be obtained. These are known with sufficient accuracy from the preliminary plots.

[14] R. K. Young. *Orbit of the Spectroscopic Binary 2 Sagittae. Journal R. A. S. C.*, vol. *11*, p. 131, 1917.

[15] *Astrophysical Journal*, vol. *40*, p. 282, 1914.

Write equation (1a) in the form

$$\rho = V + \frac{dz}{dt} = V + K\,e \cos\omega + K \cos(v + \omega) = G + K \cos(v + \omega) \qquad (17)$$

where ρ represents the observed radial velocity.

Then $(G + K)$ is the maximum, $(G - K)$ the minimum value of the velocity so that G and K may be estimated at once from the freehand curve. The period is also assumed to be known. Equation (17) may then be written in the form

$$\cos(v + \omega) = \frac{\rho - G}{K} \qquad (18)$$

and the value of $(v + \omega)$ computed for each observed value of ρ.

If we subtract the corresponding values of $M + M_0$ from each of these, we shall have values of $(v - M) + (\omega - M_0)$. The second part of this expression is constant, while the first is the equation of the center in the elliptic motion. During a revolution this varies between equal positive and negative limits which depend only on the eccentricity, and are nearly proportional to it, as is shown in the following table.

Eccentricity	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90
Maximum equation of center	11°.5	23°.0	34°.8	46°.8	59°.2	72°.3	86°.4	102°.3	122°.2

If the values of $v - M + \omega - M_0$ are plotted against those of $M + M_0$, we obtain a diagram which, since it represents the relations between the mean and the true anomalies, we may call the anomaly diagram. If on this diagram a curve is drawn to represent the plotted points, half the difference between its maximum and minimum ordinates will be the greatest value of the equation of the center, from which e may be found at once by means of the table. The mean of the maximum and minimum ordinates will be the value of $\omega - M_0$. The instants when $v - M + \omega - M_0$ has this value are those of periastron and apastron passage, the former corresponding to the ascending branch of the curve, which is always the steeper. The abscissae of the corresponding points of the curve are M_0 and $M_0 + 180°$. The values of e, M_0, and ω are now known, and the remaining elements may be found at once from K and G.

According to Russell the "principal advantage of this method is that the form of the curves which give $v - M$ as a function of M depends upon e alone." For further details and an illustrative example we refer the reader to the original memoir. Up to the present time the method has not come into general use.

A similar remark applies to the graphic method proposed by Dr. K. Laves, in 1907, and, of course, to the two short methods quite recently proposed by Dr. F. Henroteau, which take advantage of the Allegheny Tables of Anomalies in a novel way. References to the original papers containing these methods are given at the end of this chapter.

DIFFERENTIAL CORRECTIONS TO THE ELEMENTS

Whatever method may be used in finding the preliminary orbit it is generally advisable to determine the correction to the elements by the method of least squares.[16] The formula derived by Lehmann-Filhés from which the coefficients for the observation equations are to be computed may be written as follows:

$$
\begin{aligned}
d\,\frac{d\zeta}{dt} &= d\,V + [\cos(v+\omega) + e\cos\omega]\,d\,K \\
&+ K\left[\cos\omega - \frac{\sin(v+\omega)\sin v}{1-e^2}(2 + e\cos v)\right] d\,e \\
&- K[\sin(v+\omega) + e\sin\omega]\,d\,\omega \\
&+ \sin(v+\omega)\,(1 + e\cos v)^2\,\frac{K\,\mu}{(1-e^2)^{3/2}}\,d\,T \\
&- \sin(v+\omega)\,(1 + e\cos v)^2\,\frac{K}{(1-e^2)^{3/2}}\,(t-T)\,d\,\mu\,. \qquad (19)
\end{aligned}
$$

[16] *Publication Allegheny Observatory*, vol. *1*, p. 33, 1908.

The advantages of applying the method of least squares to the definitive solution of spectroscopic binary stars have been clearly stated by Schlesinger in this paper. "The problem," he says, "involves the evaluation of five quantities (six if the period be included) that are so interwoven as to make their separate determination a matter of some difficulty. Herein lies the chief advantage for this case of the method of least squares; for it enables us to vary all of the unknowns *simultaneously* instead of one or two at a time. . . . Further, the method of least squares not only has the advantage of greater accuracy and of telling us how reliable our results are, but it eliminates from the computations any personal bias or arbitrary step . . . the method should be used in almost every case where the elements are not avowedly provisional." Not all computers are so enthusiastic as to the method. Judgment must of course be exercised in all orbit work as to whether the data at hand warrant anything beyond the computation of purely provisional elements. In spectroscopic binary orbits, for example, such factors, among others, as the number and quality of the plates and their distribution over the velocity curve, the character of the star's spectrum, and the character of the orbit must be considered in making this decision.

In practice the period is almost always assumed to be known with accuracy and the last term of the equation is omitted.

To facilitate the computation Schlesinger has transformed this equation as follows:

Put

$$\alpha = 0.452 \sin v \, (2 + e \cos v)$$

$$\beta = \frac{(1 + e \cos v)^2}{(1 + e)^2}$$

$$\Gamma = d\,V + e \cos \omega \, d\,K + K \cos \omega \, d\,e - K\,e \sin \omega \, d\,\omega$$

$$\kappa = d\,K$$

$$\pi = -\,K\,d\,\omega$$

$$\epsilon = -\,K \frac{2.21}{1 - e^2}\,d\,e$$

$$\tau = K\,\mu\,\sqrt{\frac{1+e}{1-e}} \cdot \frac{1}{1-e}\,d\,T$$

$$m = -\,K\sqrt{\frac{1+e}{1-e}} \cdot \frac{1}{1-e}\,d\,\mu\,,\ \text{and } u = (v + \omega).$$

Then the equations of condition take the form

$$d\,\frac{d\zeta}{dt} = \Gamma + \cos u \cdot \kappa + \sin u \cdot \pi + \alpha \sin u \cdot \epsilon + \beta \sin u \cdot \tau + \beta \sin u \cdot (t - T)m. \qquad (20)$$

The quantities α and β can be tabulated once for all and such a tabulation is given by Schlesinger (*loc. cit.*) so arranged "as to render the normal equations homogeneous and to enable all multiplications to be made with Crelle's tables without interpolation." If this notation is used, the computer should have these tables at hand.[17]

When both spectra are visible on the plates, the orbits for the two components with respect to the center of mass may be determined separately. It is obvious that the two sets of values of V, e, T, and P must be identical, the values of ω must differ by 180°, while the two values for K depend upon

[17] Dr. R. H. Curtiss has shown that this formula and therefore the least squares solution can be made appreciably shorter. (*Publications Astronomical Observatory, University of Michigan*, Vol. II, p. 178, 1916).

the relative masses of the components. The preliminary elements for the two components, when independently determined, will, in general, not harmonize perfectly. To obtain the definitive values the best procedure is the one first suggested, I believe, by Dr. King.[18] It consists in combining all the observations, those for the secondary with those for the primary, into a single set of observation equations (equations of condition) and solving for one complete set of elements. If we write $\omega' = (\omega + 180°)$ and distinguish the values of K for the two components by writing K_1 and K_2 respectively, the equations in the notation of (19) assume the form

$$
\begin{aligned}
d\frac{d\zeta}{dt} &= dV + [\cos(v+\omega) + e\cos\omega]dK_1 + [\cos(v+\omega') + e\cos\omega']\,dK_2 \\
&+ \left[\left\{\cos\omega - \frac{\sin(v+\omega)\sin v}{1-e^2}(2+e\cos v)\right\}K_1\right. \\
&\qquad \left. + \left\{\cos\omega' - \frac{\sin(v+\omega')\sin v}{1-e^2}(2+e\cos v)\right\}K_2\right]de \\
&- \left[\left\{\sin(v+\omega) + e\sin\omega\right\}K_1 + \left\{\sin(v+\omega') + e\sin\omega'\right\}K_2\right]d\omega \\
&+ \left[\sin(v+\omega)(1+e\cos v)^2 K_1\right. \\
&\left. + \sin(v+\omega')(1+e\cos v)^2 K_2\right]\frac{M}{(1-e^2)^{3/2}}\,dT, \qquad (21)
\end{aligned}
$$

the value of the period being assumed to require no correction.

Since K_2 does not affect the residuals of the primary component, nor K_1 those of the secondary, the terms dK_2 and dK_1 disappear from the equations representing the residuals from the primary and secondary curves, respectively.

SECONDARY OSCILLATIONS

When the orbit of a spectroscopic binary star has been computed and the theoretical velocity curve drawn, it is sometimes found that the observed normal places are so distributed

[18] See Harper's paper, in *Publications of the Dominion Observatory*, Vol. 1, p. 327, 1914. Dr. Paddock independently developed an equivalent equation. *Lick Observatory Bulletin*, Vol. 8, pp. 156, 157, 1915.

with respect to the curve representing simple elliptic motion as to suggest that a secondary oscillation is superimposed upon it. The question is whether this grouping arises from some source of error in the measurement of the spectrograms, from erroneous values of one or more of the orbit elements, or from a real oscillation such as might be produced, for example, by the presence of a third body in the system. This question has been discussed by Schlesinger and Zurhellen, and later by Paddock. Schlesinger and his associates at the Allegheny Observatory have shown that 'the blend effect' caused by the overlapping of the absorption lines of the two component

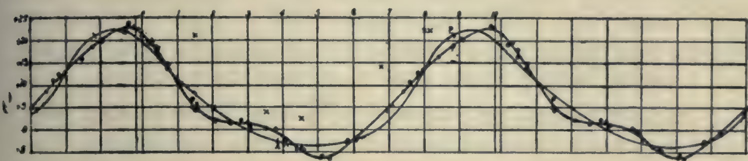

FIGURE 10. Radial Velocity Curve of ζ *Geminorum*. The secondary oscillation is probably real

spectra "may produce such an apparent oscillation." They have also shown that it may be produced by chance errors in the velocities derived from the different lines of the spectrum, and they are convinced that a critical analysis of the data will dispose of a considerable percentage of cases wherein secondary oscillations have been suspected. The possibility of a real secondary oscillation must of course be recognized, and when a full analysis has shown that such an oscillation is present, additional terms may be introduced into the equations of condition to represent it upon the assumption that it is produced by a third body revolving in a circular orbit about one of the other two components. In the cases that have arisen thus far this simple assumption has yielded a satisfactory representation of the data, though it is apparent that there is no reason for limiting such additional bodies to circular orbits.

Let T' represent the time when the secondary curve crosses the primary from below, K' the semi-amplitude of the secondary oscillation, m' the ratio of the principal period to that of the secondary oscillation, assumed to be known (it is generally taken to be an integer), and put $u' = m'\mu (t - T')$, $\tau' = - m' \mu K' dT', \kappa' = dK'$; then the additional terms required in equation (20) are

$$+ \sin u' \cdot \kappa' + \cos u' \cdot \tau' . \qquad (22)$$

For a more complete discussion of secondary oscillations the reader is referred to the articles cited above.

ORBITS WITH SMALL ECCENTRICITY

In a circular orbit the elements T and ω obviously have no significance, and when the eccentricity is very small they become practically indeterminate by the geometrical or graphical methods which have here been described. Further, if approximate values are assumed, it is impossible to find corrections to both elements from the same least squares solution because the coefficients for the differential corrections will be nearly or quite equal. Some computers have overcome this difficulty by assuming the preliminary value of ω as final, and determining corrections to T, but this is hardly a solution of the problem. In such orbits the analytic method possesses great advantages, as has been shown by several investigators, notably Wilsing and Russell, Zurhellen, and Plummer. Paddock has quite recently examined the question in great detail, extending some of the earlier developments and adapting them for computation. A full account of these methods would require more space than is available here, and it has seemed best to refer the reader to the original papers.

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CHAPTER VII

ECLIPSING BINARY STARS

We have seen that one of the first binary systems to be discovered with the spectrograph was *Algol* (*β Persei*), long known as a variable star. There are other stars whose light varies in the same peculiar manner as that of *Algol*; that is, while it remains sensibly constant at full brightness the greater part of the time, at regular intervals it fades more or less rapidly to a certain minimum. It may remain constant at this minimum for a short time and then recover full brightness, or the change may be continuous. In either case the entire cycle of change is completed in a small fraction of the time of constant light between the successive minima.

The hypothesis that in every such case the star, as viewed from the Earth, undergoes a total, annular, or partial eclipse, the eclipsing body being a relatively dark star revolving with the other about a common center of gravity, completely accounts for the observed facts and has been proved to be correct not only in the one instance, *Algol*, but also in that of every Algol-type variable which has been investigated with the spectrograph. Undoubtedly it is the correct explanation for all stars of this type; they are all binary systems.

Unless the darker star is absolutely non-luminous, there should be a second minimum when the bright star passes between it and the Earth, the relative depth of the two minima depending upon the relative intensity of the light of the two stars and upon their relative areas. Such a secondary minimum has been observed in *β Lyrae* and in this star the light is not quite constant at any phase, either maximum or minimum. There is now no doubt but that this star and others like it are also binary systems.

It was formerly thought that a distinction could be drawn between variable stars of the type of *Algol* and those of the type of *β Lyrae*, but measures with sensitive modern photometers, such as the selenium-cell, the photo-electric cell, and the sliding-prism polarizing photometers, and measures of extra-focal images on photographic plates have attained such a degree of accuracy that a variation considerably less than one-tenth of a magnitude can be detected with certainty; and it now appears that *Algol* itself not only has a slight secondary minimum but that its light is not quite constant at maximum. The distinction, therefore, breaks down and we may regard all the stars of these two types as members of a single class, calling them *eclipsing binaries* or *eclipsing variables* according to the point of view from which we take up their investigation. There are, in all, more than 150 eclipsing binary stars known at the present time and a large percentage of them are too faint to photograph with our present spectrographic equipment. It is therefore a matter of great interest to inquire what information, if any, as to the orbits of these systems can be derived from their light curves, the curves, that is, which are constructed by taking the observed stellar magnitudes as ordinates and the corresponding times as abscissae.

Professor E. C. Pickering[1] made an investigation of the orbit of *Algol* on the basis of its light curve as early as 1880, and showed that a solution of such orbits was possible if certain reasonable assumptions—for example, that the two stars are spherical with uniformly illuminated disks and move in circular orbits—were granted. The subject was resumed by him later, and was taken up also by Harting, Tisserand, A. W. Roberts, and others. But the most complete investigation so far is that begun at Princeton University, in 1912, by Russell and Shapley, the theoretical part being contributed, mainly, by the former, the application to particular systems, mainly, by the latter. The present chapter will be based entirely upon this investigation.

In the most general case the problem is an extremely complicated one, for the orbits must be regarded as elliptical with planes inclined at a greater or less angle to the line of sight,

[1] Dimensions of the Fixed Stars, with special reference to Binaries and variables of the Algol Type. *Proceedings American Academy of Arts and Sciences*, vol. *16*, p. 1, 1881.

and the two components as ellipsoids, the longest diameter of each being directed toward the other star. Moreover, the disks may or may not be uniformly illuminated; they may be darker toward the limb, as our own Sun is, the degree of darkening depending upon the depth and the composition of the enveloping atmosphere, and the side of each which receives the radiation from the other may be brighter than the opposite side. The complete specification of an eclipsing binary system therefore requires a knowledge of at least thirteen quantities, which in Russell's notation, are as follows:

Orbital Elements		Eclipse Elements	
Semi-major axis	a	Radius of larger star	r_1
Eccentricity	e	Radius of smaller star	r_2
Longitude of periastron	ω	Light of larger star	L_1
Inclination	i	Light of smaller star	L_2
Period	P	and at least 3 constants defining the amount of elongation, of darkening at the limb, and of brightening of one star by the radiation of the other.	
Epoch of principal conjunction	t_0		

The longitude of the node must remain unknown, as there is no hope of telescopic separation of any eclipsing pair.

The value of a in absolute units can be found only from spectroscopic data. In the absence of these, it is desirable to take a as an unknown but definite unit of length, and express all other linear dimensions in terms of it. Similarly, the absolute values of L_1 and L_2 can be determined only if the parallax of the system is known. But in all cases the combined light of the pair, $L_1 + L_2$, can be taken as the unit of light and the apparent brightness at any time expressed in terms of this. This leaves the problem with eleven unknown quantities to be determined from the photometric measures. Of these, the period is invariably known with a degree of accuracy greatly surpassing that attainable for any of the other elements, and the epoch of principal minimum can be determined, almost independently of the other elements, by inspection of the light-curve. Of the remaining elements, the constants expressing ellipticity and 'reflection' may be derived from the observed brightness between eclipses. These effects are often so small as to be detected only by the most refined observations. The question of darkening toward the limb may well be set aside until the problem is solved for the case of stars which appear as uniformly illuminated disks.

This leaves us with six unknowns. Fortunately, systems of such short period as the majority of eclipsing variables have usually nearly circular orbits (as is shown both by spectroscopic data and by the position of the secondary minimum). The assumption of a circular orbit is therefore usually a good approximation to the facts, and often requires no subsequent modification.

Russell's papers discuss first the simplified problem:

Two spherical stars, appearing as uniformly illuminated disks, and revolving about their common center of gravity in circular orbits, mutually eclipse one another. It is required to find the relative dimensions and brightness of the two stars, and the inclination of the orbit, from the observed light-curve.

Even in this form four different cases are presented, depending upon the character of the photometric data and the completeness of the observations; only one minimum or both minima may have been adequately observed, and these may or may not show a constant phase. To illustrate the method of investigation, I shall here present (by permission, almost entirely in Russell's own words) the two simplest cases, namely, those in which a constant phase has been observed in one or both minima and in which the eclipses are therefore either total or annular, and shall refer the reader who wishes to pursue the subject to the original memoirs.

We may assume P and t_0 as already known. If the radius of the relative orbit is taken as the unit of length, and the combined light of the two stars as the unit of light, we have to determine four unknown quantities. Of the various possible sets of unknowns, we will select the following:

Radius of the larger star	r_1
Ratio of radii of the two stars	k
Light of the larger star	L_1
Inclination of the orbit	i

The radius of the smaller star is then $r_2 = kr_1$, and its light, $L_2 = 1 - L_1$. It should be noticed that, with the above definitions, k can never exceed unity, but L_2 will exceed L_1 whenever the smaller star is the brighter (which seems to be the fact in the majority of observed cases).

We will suppose that we have at our disposal a well-determined 'light-curve', or more accurately, magnitude-curve. . . . From this we can pass at once to the intensity-curve, giving the actual light-intensity l as a function of the time, by means of the equation

$$\log l = 0.4(m_0 - m), \qquad (1)$$

where m_0 is the magnitude during the intervals of constant light between eclipses (which is determined with relatively great weight by the observations during these periods and, like P, may be found once for all before beginning the real solution). This of course expresses l in terms of our chosen unit $L_1 + L_2$.

Such a magnitude-curve or intensity-curve will in general show two depressions, or 'minima', corresponding to the mutual eclipses of the two components. Under the assumed conditions, it is well known:

1. That the two minima will be symmetrical about their middle points' and that these will be separated by exactly half the period.

2. If the eclipse is total or annular, there will be a constant phase at minimum during which the magnitude- or intensity-curve is horizontal; but if the eclipse is only partial, this will not be the case.

3. The two minima will be of equal duration, but usually of unequal depth. At any given phase during one minimum one of the stars will eclipse a certain area of the apparent disk of the other. Exactly half a period later, at the corresponding phase during the other minimum, the geometrical relations of the two projected disks will be the same, except that now the second star is in front, and eclipses an equal area—though not an equal proportion—of the disk of the first. The intensity-curves for the two minima must therefore differ from one another only as regards their vertical scales, which will be in the ratio of the surface intensities of the two stars.

4. The deeper (primary) minimum corresponds to the eclipse of the star which has the greater surface intensity by the other. Whether this is the larger or smaller star must be determined by further investigation.

Suppose that at any time during the eclipse of the smaller star by the larger the fraction α of its area is hidden. The light received from the system at this moment will be given by the equation

$$l_1 = 1 - \alpha L_2. \qquad (2)$$

Half a period later, an equal area of the surface of the larger disk, and hence the fraction $k^2\alpha$ of its whole area will be eclipsed. The observed light will then be

$$l_2 = 1 - k^2\alpha L_1. \qquad (3)$$

Since $L_1 + L_2 = 1$, we find at once from these equations

$$(1 - l_1) + \frac{1 - l_2}{k_1} = \alpha. \qquad (4)$$

If both minima have been observed and show constant phases, the eclipse of the smaller star by the larger is total and the other eclipse annular; in both cases an area equal to the whole area of the smaller star is obscured; that is, $\alpha = 1$. If λ_1 and λ_2 are the values of the observed intensities during the constant phase at the two minima, we have by (4)

$$k^2 = \frac{1 - \lambda_2}{\lambda_1}. \qquad (5)$$

Moreover, by (2), $\lambda_1 = 1 - L_2 = L_1$. The brightness of the two stars and the ratio of their radii, are thus determined, leaving only r_1 and i to be found.

There are, however, two solutions with different values of k according as we regard the principal or secondary minimum as total. We shall see later how we may distinguish the correct solution in a given instance.

[In case only the primary minimum has been observed], if the observed minimum intensity is λ and we assume that the observed eclipse is total, we have from (2), $L_2 = 1 - \lambda$; if annular, (3) gives $k^2L_1 = 1 - \lambda$. In either case, for any other value l of the observed intensity,

$$\alpha = \frac{1 - l}{1 - \lambda}. \qquad (6)$$

We thus know α as a function of the time, and from this have to determine k, r_1, and i.

Take the center of the larger star as origin, and let θ be the true longitude of the smaller star in its orbit, measured from inferior conjunction. Then [2]

$$\theta = \frac{2\pi}{P}(t - t_0). \qquad (8)$$

From the light-curve and (6) we can find the value of α for any value of θ, or vice versa. Now α, which is the fraction of the area of the smaller disk which is eclipsed at any time, depends on the radii of the two disks, and the apparent distance of their centers, but only on the ratios of these quantities (being unaffected by increasing all three in the same proportion). If δ is the apparent distance of centers, we have therefore

$$\alpha = f\left(\frac{r_2}{r_1}, \frac{\delta}{r_1}\right) = f\left(k, \frac{\delta}{r_1}\right),$$

[2] For convenience, I have preserved the equation numbers as given in Russell's original paper. His equation (7) is omitted because it does not relate to the cases here discussed.

where f is a function, the details of calculation of which will be discussed later.

For any given value of k we may invert this function, and write

$$\frac{\delta}{r_1} = \phi(k, \alpha). \qquad (9)$$

This function, or some equivalent one, may be tabulated once for all for suitable intervals of k and α.

By the geometry of the system, we have

$$\delta^2 = \sin^2\theta + \cos^2 i \cos^2\theta = \cos^2 i + \sin^2 i \sin^2\theta, \qquad (10)$$

whence

$$\cos^2 i + \sin^2 i \sin^2\theta = r_1^2\left\{\phi(k,\alpha)\right\}^2. \qquad (11)$$

Now let α_1, α_2, α_3 be any definite values of α and θ_1, θ_2, θ_3 the corresponding values of θ (which may be found from the light-curve). Subtracting the corresponding equations of the form (11) in pairs, and dividing one of the resulting equations by the other, we find

$$\frac{\sin^2\theta_1 - \sin^2\theta_2 = \left\{\phi(k,\alpha_1)\right\}^2 - \left\{\phi(k,\alpha_2)\right\}^2}{\sin^2\theta_2 - \sin^2\theta_3 = \left\{\phi(k,\alpha_2)\right\}^2 - \left\{\phi(k,\alpha_3)\right\}^2} = \psi(k, \alpha_1, \alpha_2, \alpha_3). \qquad (12)$$

The first member of this equation contains only known quantities. The second, if α_1, α_2, and α_3 are predetermined, is a function of k alone. If this function is tabulated, the value of k in any given case can be found by interpolation, or graphically. The equations (11) can then be used to find r_1 and i.

A theoretical light-curve may then be found, which passes through any three desired points on each branch of the observed curve (assumed symmetrical). These points may be chosen at will by altering the values of α_1, α_2, and α_3. In practice it is convenient to keep α_2 and α_3 fixed, so that ψ becomes a function of k and α_1 only, and may be tabulated for suitable intervals in these two arguments. This has been done in Table II, in which α_2 is taken as 0.6 and α_3 as 0.9. If $A = \sin^2\theta_2$, $B = \sin^2\theta_2 - \sin^2\theta_3$, (12) may be written

$$\sin^2\theta_1 = A + B\psi(k, \alpha_1). \qquad (13)$$

The points a and b on the light-curve corresponding to α_2 and α_3, together with the point corresponding to any one of the tabular values of α_1, then give a determination of k. By taking a suitably weighted mean of these values of k, a theoretical light-curve can be obtained which passes through the points a and b, and as close as possible to the others. By slight changes in the assumed positions of a and b (*i.e.*, in the corresponding values of θ, or of $t - t_0$), it is possible with little labor to obtain a theoretical curve which fits the whole course of the observed curve almost as well as one determined by least squares. The criterion of this is that the parts of the observed curve below b (near totality), between a and b, and above a (near the beginning or end of eclipse) shall give the same mean value of k. The individual determinations of k are of very different weight. Between a and b (that is for values of α_1 between 0.6 and 0.9) ψ changes very slowly with k. At the beginning and end of the eclipse the stellar magnitude changes very slowly with the time, and hence, by (13), with ψ. The corresponding parts of the curve are therefore ill adapted to determine k. For the first approximation it is well to give the values of k derived from values of α_1 between 0.95 and 0.99, and between 0.4 and 0.2, double weight (provided the corresponding parts of the curve are well fixed by observation). The time of beginning or end of eclipse cannot be read with even approximate accuracy from the observed curve and should not be used at all in finding k. The beginning or end of totality may sometimes be determined with fair precision, but does not deserve as much weight as the neighboring points on the steep part of the curve. If further refinement is desired, it can most easily be obtained by plotting the light-curve for two values of k and comparing with a plot of the observations. This will rarely be necessary.

When once k is given, the determination of the light-curve is a very easy matter. For each tabular value of α_1, equation (13) gives θ_1, and hence $t_1 - t_0$. The values of the stellar magnitude m corresponding to given values of α_1 are already available, having been used in the previous work. The light-curve may thus be plotted by points in a few minutes.

After a satisfactory light-curve has been computed, we may proceed to determine the remaining elements. Let θ' and θ'' be the values corresponding to the beginning of eclipse ($\alpha_1 = 0$) and to the beginning of totality ($\alpha_1 = 1$). Then by (13)

$$\sin^2\theta' = A + B\psi(k, 0) \text{ and } \sin^2\theta'' = A + B\psi(k, 1).$$

These computed values are more accurate than those estimated from the free-hand curve drawn to represent the observations. At the first of these epochs $\delta = r_1 + r_2$, and at the second $\delta = r_1 - r_2$. We have then, by (10)

$$r_1^2(1 + k)^2 = \cos^2 i + \sin^2 i \sin^2\theta',$$
$$r_1^2(1 - k)^2 = \cos^2 i + \sin^2 i \sin^2\theta'',$$

whence

$$4k \cot^2 i = (1 - k)^2 \sin^2\theta' - (1 + k)^2 \sin^2\theta'',$$
$$4kr_1^2(1 + \cot^2 i) = \sin^2\theta' - \sin^2\theta''.$$

Introducing A and B, we have

$$4k \cot^2 i = -4kA + B\{(1-k)^2\psi(k, 0) - (1+k)^2\psi(k, 1)\},$$
$$4kr_1^2 \operatorname{cosec}^2 i = B\{\psi(k, 0) - \psi(k, 1)\}.$$

The coefficients are functions of k alone, and may be tabulated. It is most convenient for this purpose to put the equations in the form

$$\left.\begin{aligned} r_1^2 \operatorname{cosec}^2 i &= \frac{B}{\phi_1(k)}, \\ \cot^2 i &= \frac{B}{\phi_2(k)} - A, \end{aligned}\right\} \quad (14)$$

as in this way we obtain functions whose tabular differences are comparatively smooth (which is not true of their reciprocals). With the aid of these functions the elements may be found as soon as A and B are known. If $\frac{B}{A} < \phi_2(k)$ the computed value of $\cot i$ is imaginary and the solution is physically impossible. It is therefore advisable to apply this test to the preliminary values of A, B, and k, and, if necessary, to adjust them so that the solution is real. The limiting condition is evidently $\cot i = 0$, corresponding to central transit.

The geometrical elements of the system are now determined. We are still in doubt, however, whether the principal eclipse is total or annular. This can be determined only by consideration of the secondary minimum. The intensities during constant phase at the two minima are connected by the relation $k^2\lambda_1 + \lambda_2 = 1$. If the intensity at principal minimum is λ_p, that at the secondary minimum will be $1 - k^2\lambda_p$ if the principal eclipse is total, and $\frac{1-\lambda_p}{k^2}$ if it is annular. The first of these expressions is always positive and less than unity. The second exceeds unity if $1 - \lambda_p > k^2$. The assumption of total eclipse at principal minimum leads therefore in all cases to a physically possible solution. That of an annular eclipse does so only if $1 - \lambda_p$ is not greater than k^2. Otherwise the computed brightness of the smaller star is negative. The brightness at secondary minimum will be greater than at the primary by $1 - \lambda_p(1 + k^2)$ if the primary eclipse is total, and $\frac{1}{k^2}[1 - \lambda_p(1 + k^2)]$ if it is annular. The latter hypothesis therefore gives rise to the shallower minimum. In many cases it may be impossible to decide between the two without actual observations of the secondary phase. The computed depth of secondary minimum may, however, be so great that it is practically certain that it would sometimes have been observed if it really existed. The corresponding hypothesis should then be rejected. If $\lambda_p(1 + k^2)$ is nearly unity, the primary and secondary minimum, on both hypotheses, must be of nearly equal depth. This can occur only if $\lambda_p < \frac{1}{2}$; that is, if the depth of minimum is less than 0.75 mag. In such a case it is probable that the period is really twice that so far assumed, that the two stars are of equal surface brightness, and that two sensibly equal eclipses occur during each revolution. The true values of θ are therefore half those previously computed with the shorter period. If the determination of k is repeated on this basis, and the equation $\lambda_p(1 + k^2) = 1$ is still approximately satisfied this solution may be adopted.

Such a system presents a specialized example of [the case], when both primary and secondary minima have been observed and show a constant phase. In this case, by (5), $k^2 = \dfrac{1 - \lambda_2}{\lambda_1}$ where λ_1 corresponds to the total eclipse, which, so far as we yet know, may occur at either minimum. As before we begin by finding from the light-curve the values of $\sin^2 \theta$ corresponding to given values of α_1. From a few of these, by the method already described, an approximate value of k may be obtained which is sufficient to show which of the values given by (5) on the two possible hypotheses is the correct one.

We have next to find the light-curve which gives the best representation of the observations consistent with the value of k given by (5). The form of the light-curve now depends only on the constants A and B in the equation

$$\sin^2 \theta_1 = A + B\psi(k, \alpha_1). \qquad (13)$$

Approximate values of these constants may be derived as above from the values of $\sin^2 \theta$ when $\alpha = 0.6$ and 0.9. These may be improved by trial and error, which will be aided by plotting the resulting light-curves along with the observations, and, if the data warrant it, may finally be corrected by least squares. When satisfactory values of A and B have been determined, the final light-curve may be computed by (13), and the elements by (14), as [before], except that here there is no uncertainty as regards the nature of the principal eclipse.

In review of the foregoing, it may be remarked that the method of solution is direct and simple. It involves a very moderate amount of numerical work, of which the greater part—namely, the determination of the values of the magnitude, time, and position in orbit (θ) corresponding to different percentages of obscuration (α)—requires no modification during the successive approximations. The light-curve may be found without the necessity of computing the elements, and with two or three trials may be determined so as to represent the whole course of the observations, making the laborious solution by least squares superfluous except in the case of observations of unusual precision. Such a solution itself is much simplified if the constants defining the light-curve, instead of the elements of the system, are treated as the fundamental unknowns, as the coefficients of the equations of condition may be easily found graphically with the aid of data already computed. The elements may be found, at any stage of the process, by a few moments' calculation, from the constants defining the light-curve.

Russell's paper contains a number of tables, of which the four directly applicable to the cases of total or annular eclipses which have been discussed are reprinted here. His Table I, tabulating the function given in equation (9), is omitted, though it is fundamental, because it is used, so far as we are at present concerned, only in constructing the subsequent tables.

Table II contains the function $\psi(k, \alpha_1)$ defined by the equation

$$\psi(k, \alpha_1) = \frac{\left\{1 + kp(k, \alpha_1)\right\}^2 - \left\{1 + kp(k, \alpha_2)\right\}^2}{\left\{1 + kp(k, \alpha_2)\right\}^2 - \left\{1 + kp(k, \alpha_3)\right\}^2},$$

(where $\alpha_2 = 0.6$ and $\alpha_3 = 0.9$), which is used in determining k in the case of total eclipse. The uncertainty of the tabular quantities does not exceed one or two units of the last decimal place, except for the larger values of ψ, corresponding to values of α_1 less than 0.3, for which the actual errors may be greater, but are not more serious in proportion to the whole quantity tabulated.

Table II*a* contains the functions

$$\phi_1(k) = \frac{4k}{\psi(k, 0) - \psi(k, 1)} \text{ and } \phi_2(k) = \frac{4k}{(1-k)^2\psi(k, 0) - (1+k)^2\psi(k, 1)},$$

which are useful in determining the elements in the case of total eclipse.

Table A gives the *loss* of light $(1 - \lambda)$, corresponding to a given change Δm in stellar magnitude. For a difference of magnitude greater than 2.5, the loss of light is 0.9000+one-tenth of the tabular value for $\Delta m - 2^{\mathrm{m}}.5$. Table B gives the values of $\theta - \sin\theta$ for every 0.01 of θ (expressed in circular measure), and saves much labor in computing the values of $\sin\theta$ corresponding to a given interval from minimum.

TABLE II. *For Use in Case of Total Eclipse. Values of* ψ (k, α_1)

α_1	k=1.00	0.90	0.80	0.70	0.60	0.50	0.40	0.30	0.20	0.10	0.00
0.00	+9.464	+7.478	+6.200	+5.279	+4.556	+3.984	+3.503	+3.104	+2.755	+2.454	+2.199
.02	8.095	6.457	5.373	4.606	4.000	3.504	3.106	2.768	2.478	2.216	2.000
.05	7.042	5.616	4.704	4.047	3.534	3.118	2.777	2.488	2.241	2.017	1.829
0.10	+5.759	+4.625	+3.895	+3.364	+2.960	+2.627	+2.358	+2.131	+1.934	+1.754	+1.603
.15	4.755	3.839	3.248	2.826	2.504	2.240	2.024	1.841	1.682	1.537	1.412
.20	3.906	3.184	2.712	2.374	2.110	1.898	1.726	1.581	1.453	1.336	1.235
0.25	+3.158	+2.600	+2.232	+1.969	+1.760	+1.591	+1.453	+1.344	+1.242	+1.146	+1.070
.30	2.522	2.088	1.803	1.603	1.443	1.314	1.205	1.115	1.039	0.968	0.911
.35	1.979	1.641	1.425	1.276	1.157	1.061	0.982	0.911	0.854	0.797	.756
0.40	+1.490	+1.245	+1.087	+0.978	+0.894	+0.825	+0.770	+0.721	+0.675	+0.633	+0.604
.45	1.040	0.881	0.777	.705	.649	.603	.566	.530	.501	.473	.453
.50	0.648	.555	.491	.451	.418	.392	.370	.348	.331	.314	.302
0.55	+0.300	+0.258	+0.233	+0.217	+0.202	+0.191	+0.181	+0.171	+0.164	+0.156	+0.151
.60	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
.65	− .258	−0.231	−0.214	−0.202	− .191	−0.181	−0.174	−0.167	−0.160	−0.156	−0.152
0.70	−0.480	−0.435	−0.408	−0.387	−0.369	−0.354	−0.344	−0.331	−0.320	−0.314	−0.306
.75	− .660	− .613	− .584	− .558	− .539	− .522	− .508	− .494	− .483	− .475	− .465
.80	− .805	− .765	− .738	− .717	− .700	− .684	− .670	− .659	− .647	− .639	− .632
0.85	−0.922	−0.893	−0.877	−0.863	−0.854	−0.843	−0.833	−0.825	−0.818	−0.812	−0.808
.90	−1.000	−1.000	−1.000	−1.000	−1.000	−1.000	−1.000	−1.000	−1.000	−1.000	−1.000
.95	−1.045	−1.085	−1.112	−1.134	−1.152	−1.166	−1.179	−1.190	−1.203	−1.214	−1.226
0.98	−1.0625	−1.126	−1.176	−1.220	−1.256	−1.284	−1.308	−1.329	−1.350	−1.369	−1.391
.99	−1.0643	−1.139	−1.199	−1.250	−1.293	−1.328	−1.362	−1.390	−1.419	−1.444	−1.471
1.00	−1.0650	−1.155	−1.231	−1.297	−1.354	−1.402	−1.445	−1.484	−1.525	−1.556	−1.596

TABLE IIa

For Computing the Elements in the Case of Total Eclipse

k	$\phi_1(k)$	$\phi_2(k)$
1.00	0.380	0.939
0.95	.401	.894
.90	.417	.848
0.85	0.427	0.802
.80	.431	.755
.75	.431	.709
0.70	0.427	0.663
.65	.419	.617
.60	.406	.572
0.55	0.390	0.527
.50	.371	.482
.45	.349	.436
0.40	0.323	0.390
.35	.294	.345
.30	.262	.298
0.25	0.226	0.250
.20	.187	.202
.15	.145	.153
0.10	0.100	0.103
.05	.052	.052
.00	.000	.000

TABLE A

Loss of Light Corresponding to an Increase Δm in Stellar Magnitude

Δm	0	1	2	3	4	5	6	7	8	9
0.0	0.0000	0.0092	0.0183	0.0273	0.0362	0.0450	0.0538	0.0624	0.0710	0.0795
.1	.0880	.0964	.1046	.1128	.1210	.1290	.1370	.1449	.1528	.1605
.2	.1682	.1759	.1834	.1909	.1983	.2057	.2130	.2202	.2273	.2344
.3	.2414	.2484	.2553	.2621	.2689	.2756	.2822	.2888	.2953	.3018
.4	.3082	.3145	.3208	.3270	.3332	.3393	.3454	.3514	.3573	.3632
.5	0.3690	0.3748	0.3806	0.3862	0.3919	0.3974	0.4030	0.4084	0.4139	0.4192
.6	.4246	.4298	.4351	.4402	.4454	.4505	.4555	.4605	.4654	.4703
.7	.4752	.4848	.4848	.4895	.4942	.4988	.5034	.5080	.5125	.5169
.8	.5214	.5258	.5301	.5344	.5387	.5429	.5471	.5513	.5554	.5594
.9	.5635	.5675	.5715	.5754	.5793	.5831	.5870	.5907	.5945	.5982
1.0	0.6019	0.6055	0.6092	0.6127	0.6163	0.6198	0.6233	0.6267	0.6302	0.6336
1.1	.6369	.6403	.6435	.6468	.6501	.6533	.6564	.6596	.6627	.6658
1.2	.6689	.6719	.6749	.6779	.6808	.6838	.6867	.6895	.6924	.6952
1.3	.6980	.7008	.7035	.7062	.7089	.7116	.7142	.7169	.7195	.7220
1.4	.7246	.7271	.7296	.7321	.7345	.7370	.7394	.7418	.7441	.7465
1.5	0.7488	0.7511	0.7534	0.7557	0.7579	0.7601	0.7623	0.7645	0.7667	0.7688
1.6	.7709	.7730	.7751	.7772	.7792	.7812	.7832	.7852	.7872	.7891
1.7	.7911	.7930	.7949	.7968	.7986	.8005	.8023	.8041	.8059	.8077
1.8	.8095	.8112	.8129	.8146	.8163	.8180	.8197	.8214	.8230	.8246
1.9	.8262	.8278	.8294	.8310	.8325	.8340	.8356	.8371	.8386	.8400
2.0	0.8415	0.8430	0.8444	0.8458	0.8472	0.8486	0.8500	0.8514	0.8528	0.8541
2.1	.8555	.8568	.8581	.8594	.8607	.8620	.8632	.8645	.8657	.8670
2.2	.8682	.8694	.8706	.8718	.8729	.8741	.8753	.8764	.8775	.8787
2.3	.8798	.8809	.8820	.8831	.8841	.8852	.8862	.8873	.8883	.8893
2.4	.8904	.8914	.8924	.8933	.8943	.8953	.8962	.8972	.8981	.8991
2.5	0.9000	0.9009	0.9018	0.9027	0.9036	0.9045	0.9054	0.9062	0.9071	0.9080

For values of Δm greater than 2.5, the loss of light is 0.9000 plus $\frac{1}{10}$ of the loss of light corresponding to $\Delta m - 2.5$.

TABLE B

Values of θ − Sin θ

	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.00	0.0000	0.0002	0.0013	0.0045	0.0105	0.0206	0.0354	0.0558	0.0826	0.1167
.01	.0000	.0002	.0015	.0049	.0114	.0218	.0372	.0582	.0857	.1205
.02	.0000	.0003	.0018	.0055	.0122	.0231	.0390	.0607	.0888	.1243
.03	.0000	.0004	.0020	.0060	.0131	.0244	.0409	.0632	.0920	.1283
.04	.0000	.0005	.0023	.0066	.0141	.0258	.0428	.0658	.0953	.1324
0.05	0.0000	0.0006	0.0026	0.0071	0.0151	0.0273	0.0448	0.0684	0.0987	0.1365
.06	.0000	.0007	.0029	.0078	.0161	.0288	.0469	.0711	.1022	.1407
.07	.0001	.0008	.0033	.0084	.0171	.0304	.0490	.0739	.1057	.1450
.08	.0001	.0010	.0037	.0091	.0183	.0320	.0512	.0767	.1093	.1494
.09	.0001	.0011	.0041	.0098	.0194	.0337	.0535	.0796	.1130	.1539

To illustrate Russell's method I have chosen his orbit of *W Delphini*, which is a "typical Algol variable with a deep primary minimum, showing a constant phase, and little or no

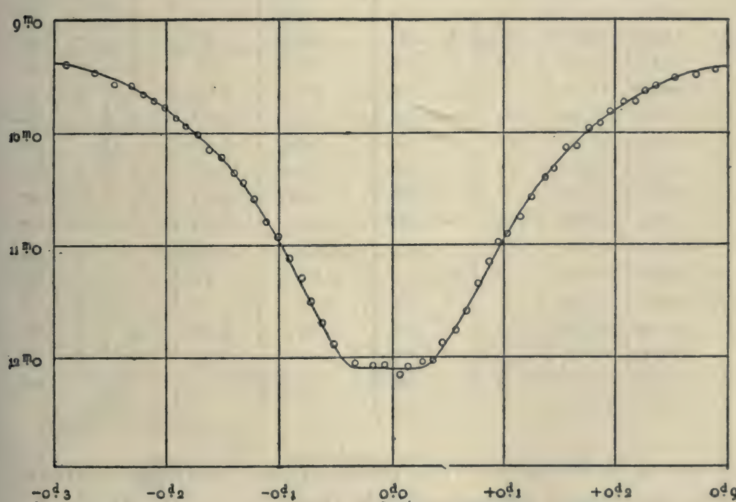

FIGURE 11. Light-Curve of the Principal Minimum of *W Delphini*

secondary minimum." Its light curve, "defined by the 500 observations by Professor Wendell, with a polarizing photometer, which are published in the Harvard Annals, 69, Part 1,"

TABLE *a*

Observed Magnitudes

Phase	Mag.	No. Obs.	O.−C.	Phase	Mag.	No. Obs.	O.−C.
$-0^d.2894$	9.41	6	$+0^m.01$	$+0^d.0560$	11.76	7	$+0^m.01$
.2637	9.49	5	+ .02	.0659	11.58	8	+ .01
.2458	9.58	5	+ .04	.0753	11.33	7	− .04
.2306	9.59	4	− .01	.0859	11.14	5	− .02
.2200	9.67	5	.00	.0937	10.97	5	− .05
.2106	9.73	8	+ .01	.1036	10.88	8	+ .02
.2007	9.79	10	.00	.1147	10.73	8	+ .05
.1911	9.88	12	+ .02	.1246	10.56	12	+ .03
.1817	9.95	10	+ .01	.1351	10.39	14	.00
.1718	10.02	8	.00	.1445	10.31	11	+ .04
.1615	10.16	17	+ .04	.1546	10.13	10	− .02
.1506	10.23	14	.00	.1641	10.10	11	+ .04
.1396	10.37	14	+ .01	.1744	9.97	10	.00
.1311	10.44	16	− .03	.1847	9.90	9	+ .02
.1212	10.59	17	− .03	.1941	9.79	9	− .02
.1121	10.78	14	+ .01	.2050	9.71	8	− .02
.1013	10.91	17	− .04	.2157	9.71	6	+ .04
.0906	11.12	14	− .01	.2242	9.63	8	+ .01
.0809	11.30	10	− .02	.2345	9.57	7	.00
.0715	11.51	12	.00	.2507	9.50	7	.00
.0617	11.69	10	.00	.2708	9.48	7	+ .03
.0509	11.88	7	.00	.2811	9.43	4	+ .02
.0313	12.05	5	− .04	.94	9.42	5	+ .02
.0169	12.08	4	− .02	1.90	9.35	5	− .05
− .0082	12.07	7	− .03	2.04	9.41	7	+ .01
+ .0060	12.16	5	+ .06	2.67	9.38	5	− .02
.0139	12.09	4	− .01	3.04	9.42	3	+ .02
.0261	12.03	5	− .07	4.04	9.44	6	+ .04
.0356	12.02	6	− .03	4.48	9.36	7	− .04
+.0460	11.87	6	− .04				

is shown in Figure 11. The observations have been combined into the normal places given in Table *a*, on the basis of a period of 4.8061 days, which was found to require no correction.

From the thirty-eight observations outside minimum we find the magnitude during constant light to be $9^{m}.395 \pm 0.009$. There is no evidence of any change during this period. With a circular orbit, the secondary minimum should occur at phase $2^{d}.40$. As none of the observations fall within $0^{d}.27$ of this, they give us no information whether such a minimum exists. The light-curve of the principal minimum is very well determined. The eclipse lasts from about $-0^{d}.28$ to $+0^{d}.28$, and there is a short constant period at the middle, of apparently a little less than one-tenth the total duration of the eclipse. The mean of the twenty observations lying within $0^{d}.02$ of the middle of eclipse gives for the magnitude at this phase $12^{m}.10 \pm 0.014$. The range of variation is therefore $2^{m}.70$, and the light-intensity at minimum 0.0832 times that at maximum. This shows at once that the eclipse is total, for if it was annular, the companion (even if perfectly dark) must cut off at least 0.917 of the light of the primary, and hence its radius cannot be less than 0.956 times that of the latter. In such a system the duration of the annular phase could not exceed $\frac{0.044}{1.956}$, or 0.022 of the whole duration of eclipse. The observed constant phase is almost five times as long as this.

The brighter star therefore gives 0.9168 of the whole light of the system, and if isolated would appear of magnitude 9.49; while the fainter but larger star which eclipses it gives out only one-eleventh as much light, and when seen alone at minimum is of magnitude 12.10.

The loss of light $(1 - l)$ at any given time, t, will be $0.9168a_1$, since a_1 is the percentage of obscuration. For a series of values of a_1 we tabulate the values of $(1-l)$ and then take from Table A the corresponding changes of magnitude and apply them to magnitude 9.4. Next, from the free-hand curve drawn to represent the data of observation, we read off the epochs t_1 and t_2 at which the magnitudes so computed are reached before and after the middle of eclipse. Half the difference of t_1 and t_2 may be taken as the interval t from the middle of eclipse to each phase and the corresponding value of θ formed from

$$\theta = \frac{2\pi}{P} t = 1.3065t,$$ where θ is expressed in radians and t in days.

With the aid of Table B $\sin\theta$ is found and then $\sin^2\theta$. These quantities are all entered in Table b.

TABLE b[3]

α_1	$1 - l$	Mag.	t_1	t_2	θ	$\sin^2\theta$	$\sin^2\theta - A$	$\psi\ (k, \alpha_1)$	k
0.0	0.0000	$9^m.400$	$-0^d.304$:	$+0^d.300$:	0.394:	0.1474:	0.1105:	+4.28:	0.56:
.1	.0917	9.505	.2540	.2515	.3304	.1050	.0681	2.64	.504
.2	.1834	9.620	.2285	.2258	.2968	.0860	.0491	1.908	.505
.3	.2750	9.749	.2075	.2030	.2681	.0702	.0333	1.290	.480
.4	.3667	9.896	.1884	.1830	.2426	.0576	.0207	0.802	.462
.5	.4584	10.066	.1682	.1644	.2173	.0462	.0193	+0.361	.36:
.6	.5500	10.266	.1486	.1470	.1931	.0369	.0000	......	....
.7	.6417	10.514	.1270	.1274	.1661	.0272	− .0097	−0.376	.64:
.8	.7334	10.835	.1070	.1048	.1381	.0190	− .0179	−0.694	.56:
.9	.8250	11.292	.0824	.0788	.1054	.0111	− .0258	−1.000	....
.95	.8709	11.624	.0655	.0624	.0836	.0071	− .0298	−1.155	.58:
.98	.8985	11.884	.0505	.0462	.0632	.0040	− .0329	−1.277	.525
.99	.9076	11.985	.0430	.0390	.0536	.0029	− .0340	−1.318	.528
1.00	.9168	12.100	− .021:	+ .019:	.026:	.0007:	− .0362:	−1.404:	.50:

[3] Russell's Computations were made with a slide-rule. Repeating them with 5-place logarithms, I obtain figures, which sometimes differ slightly from those tabulated. These differences, however, are unimportant for they produce no appreciable changes in the final elements.

From the values of t_1 and t_2 it appears that the observed curve is remarkably symmetrical, and that the actual epoch of mid-eclipse is 0.0015 days earlier than that assumed by Wendell. The times of beginning and ending of the eclipse cannot be read accurately from the curve and are marked with colons to denote uncertainty.

From the values of $\sin^2 \theta$ we have now to find k with the aid of Table II. From equation (13) we have

$$\psi\ (k,\ a_1) = \frac{\sin^2 \theta_1 - A}{B},$$

hence, if we let A be the value of $\sin^2 \theta$ when $a_1 = 0.6$ and $A - B$ its value when $a_2 = 0.9$, we may find a value of k for every tabulated value of a_1 by inverse interpolation in Table II. Thus, taking $A = 0.0369$ and $B\ (= \sin^2 \theta_2 - \sin^2 \theta_3) = 0.0258$, as given by our curve, we find for $a_1 = 0.0$ that $\psi\ (k,\ a_1) = + 4.28$:, and hence, from the first line of Table *II*, $k = 0.56$:. Colons are here used because the values of k are less accurate when the tabular differences of $\psi\ (k,\ a_1)$ are small.

The values of k are seen to be fairly accordant except for those corresponding to values a_1 near 0.6. Inspection of Table II "shows that this discrepancy may be almost removed by increasing all the values of ψ by 0.024 – which may be done by diminishing A by $0.024B$. Our new value of A is therefore 0.0363." The new set of k's are found to be discordant for values of a_1 near 0.9; "but by diminishing B by 2½ per cent. [giving $B = 0.0252$], and hence increasing all the computed values of ψ in the corresponding ratio, we obtain a third approximation of a very satisfactory character." The general mean is now $k = 0.528$.

With these final constants, $A = 0.0363$, $B = 0.0252$, $k = 0.528$, we may compute a theoretical light-curve and also the elements of the system from equation (14). Table *c* gives the second and third approximations to the value of k and the data for the final light-curve.

Plotting the magnitudes computed in Table *a* against the epochs $- 0^d.0015 \pm t$, we obtain the computed light-curve. The residuals (O. – C.) are given in the last column of Table *a*. Their average value, regardless of sign, is $0^m.020$.

TABLE c

a_1	2D Approx.		3D Approx.		Final Light-Curve				
	ψ	k	ψ	k	$\psi(0.528, a_1)$	$B\psi$	$\sin^2\theta$	$\sin\theta$	t
0.0	+4.30:	0.56:	+4.40:	0.58:	+4.100	0.1032	0.1395	0.373	0.292
.1	2.665	.512	2.73	.534	2.713	.0683	0.1046	0.324	.252
.2	1.932	.517	1.974	.538	1.949	.0491	.0854	.292	.227
.3	1.314	.500	1.344	.527	1.348	.0338	.0701	.265	.205
.4	0.826	.503	0.845	.532	0.843	.0212	.0575	.240	.185
.5	0.385	.47:	0.394	.51:	+0.400	.0101	.0464	.215	.166
.6	+0.024	.....	+0.025	.....	0.000	.0000	.0363	.191	.146
.7	−0.352	.48:	−0.360	.54:	−0.358	−.0090	.0273	.165	.127
.8	−0.670	.40:	−0.685	.51:	−0.689	−.0173	.0190	.138	.106
.9	−0.976	.....	−1.000	.....	−1.000	−.0252	.0111	.105	.081
.95	−1.131	.714	−1.157	.564	−1.162	−.0293	.0070	.084	.064
.98	−1.253	.610	−1.282	.507	−1.276	−.0322	.0041	.064	.049
.99	−1.294	.597	−1.324	.512	−1.318	−.0332	.0031	.056	.043
1.00	−1.38:	0.55:	−1.412:	0.48:	−1.389	−.0350	.0013	.036	.028

From Table II*a* we find for $k = 0.528$, $\phi_1(k) = 0.382$, $\phi_2(k) = 0.507$; whence

$$\cot^2 i = \frac{B}{\phi_2(k)} - A = 0.0133,\ \cot i = 0.115,\ i = 83^\circ\ 25'.$$

$$r_1^2 \operatorname{cosec}^2 i = \frac{B}{\phi_1(k)} = 0.0660,\ r_1^2 = 0.0652,\ r_1 = 0.256,$$

and finally

$$r_2 = kr_1 = 0.135.$$

In other words, we have, taking the radius of the orbit as unity,

Radius of larger star	0.256
Radius of smaller star	0.135
Inclination of orbit plane	83°25′
Least apparent distance of centers	0.114
Light of larger star	0.0832
Light of smaller star	0.9168
[Period of revolution	4.8061 days]

At the middle of eclipse, the larger star overlaps the other by only 0.007 of the radius of the orbit, or about one-twentieth of the radius of the smaller body, so that the eclipse is very nearly grazing. The smaller star gives off eleven times as much light as the other, and exceeds it forty-fold in surface-brightness.

The loss of light at secondary minimum should be k times the light of the fainter star, or 0.023 of that of the system. The corresponding change in stellar magnitude is 0.027, which could only be detected by refined observations.

In an earlier paper Russell gives the equations which determine the density of the components relatively to the density of the Sun in terms of the orbit elements.

Let the total mass of the system be m, that of the larger star my and that of the smaller $m(1 - y)$. If a is the semi-axis major of the orbit, we shall have $a = Km^{1/3}\ P^{2/3}$, where K is a constant depending on the units of measurement. If we choose the Sun's mass, the Sun's radius, and the day as units, then for the Earth's orbital motion $a = 214.9$, $P = 365.24$, whence $K = 4.206$.

In determining the elements of the system, we have taken a as our unit of length. The actual radius of the larger star is therefore ar_1, and its volume in terms of that of the Sun is $K^3mP^2r_1^3$, or $74.4\ mp^2r_1^3$.

Its density ρ_1 is therefore $\rho_1 = 0.01344 \frac{y}{P^2 r_1^3}$, and similarly, of the smaller star $\rho_2 = 0.01344 \frac{1 - y.}{P^2 r_2^3}$.

The actual densities can be computed only when the ratio of masses of the two stars is known. Assuming the two components of *W Delphini* to be of equal mass, the equations give

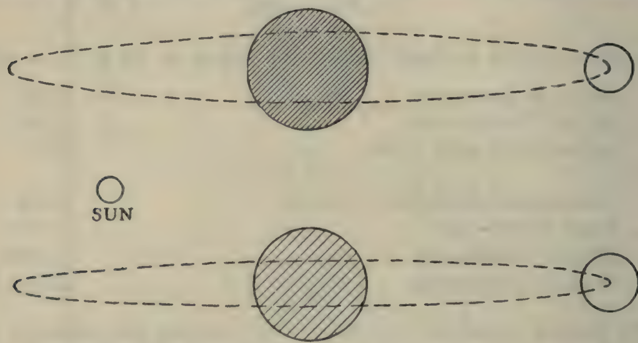

FIGURE 12. The System of *W Delphini*. The relative orbits of the bright star are shown; the upper diagram representing the elements as given in the accompanying solution, the lower, Shapley's elements on the assumption that the stars are darkened to zero at the limb. The diameters of the disks of stars and Sun are drawn on the same scale; the three bodies have the same mass, but the stars are less dense than the Sun.—From Shapley's article in *Popular Astronomy*, vol. 20, p. 572, 1912

0.118 of the Sun's density for the density of the smaller and brighter star and 0.017 for the density of the larger and fainter one.

Dr. Harlow Shapley, who was associated with Professor Russell in the theoretical investigations which have been described above, has applied the theory in a very comprehensive manner, and has published (in No. 3 of the *Contributions from the Princeton University Observatory*, 1915) "A Study of the Orbits of Eclipsing Binaries" containing orbits for ninety different systems, with a thorough discussion of the results. It is not expedient to copy his table of orbits here, for, aside from the space required, it would be necessary to extend considerably the discussion already given, in order to make all of the details of the table readily intelligible. Some general statements concerning the table and Shapley's discussion must suffice.

Since there is both theoretical and observational support for the hypothesis that the disks of eclipsing variable stars are not uniformly illuminated but are darkened toward the limb, like our Sun, by the increased absorption of their enveloping atmospheres, Shapley gives for each system at least two orbits, one for the hypothesis of uniform illumination, the other for the hypothesis of darkening toward the limb. When the nature of the data does not permit a definitive solution, several orbits are given to show the limits within which the elements may be varied. In all, more than 200 orbits are tabulated, nearly all of them computed by Shapley himself. The results seem to justify the conclusion that darkening toward the limb characterizes practically all of the stars under discussion, but the *degree* of darkening cannot as yet be determined with any certainty.

Shapley finds a distinct correlation between the range of light variation and the relative sizes of the two components; when at principal minimum the star is two magnitudes or more fainter than when at maximum, the brighter star is never the larger, and it is only rarely the larger when this range is in excess of one magnitude. On the other hand, when the range is less than 0.7 magnitudes, "there is not a single system known where the fainter star is the larger."

There is a positive indication in all but a very few cases "that the fainter star is self-luminous and in no case is it necessary to assume one component completely black. In about two-thirds of the systems the difference in brightness of the components does not exceed two magnitudes, and no observed difference is greater than four magnitudes." A large percentage of the visual binaries have a far greater difference of brightness between the two components. The fainter star, in the eclipsing systems, has always been found to be redder, and hence probably of a later class of spectrum than the bright star, whenever it has been possible to determine the relative color-index of the two components.

The great majority of the eclipsing binaries at present known belong to the spectral classes B and A. When the densities are computed, on the assumption that the mass of each component is equal to that of the Sun, it is found that the densities of stars of spectral classes B and A lie mainly between 0.02 and 0.20 of the density of the Sun. The range in density of the small number of eclipsing binaries of spectral classes F, G, and K is much greater, exceeding the density of the Sun in one system (W *Ursae Majoris*, Class G) and falling below 0.0001 of the Sun's density in two instances, while in only two does it fall within the limits 0.02 and 0.20.

Definite values for the eccentricity of the orbit could be determined in only about a dozen systems, the maximum value being 0.138 (for R *Canis Majoris*); a number of systems are known to be practically circular, but in most systems the evidence on this point is insufficient. However it may safely be said that the departure from a circular orbit is never very great.

Whenever there was evidence of ellipticity of the disks of the two components, caused by their mutual attraction, it was found that "the degree of the elongation depended directly upon the relative distance separating the two stars", and that the numerical values were in good agreement with Darwin's theoretical values for homogeneous incompressible fluids. The ellipticity has rarely been measured when the distance between the components equals or exceeds the sum of their radii.

Russell and Shapley have also studied the distribution of the eclipsing binaries in space and reference to their results will be made in a later chapter.

Stebbins, among others, has reminded us that every short period spectroscopic binary star would be an eclipsing variable to an observer in the plane of its orbit, and he has drawn the corollary, that probably a number of eclipsing variables with relatively small range in brightness exist among the known spectroscopic binaries. Acting upon this conclusion, he took up the examination of several such systems with the sensitive selenium-cell photometer which he had developed, and was able to show that the first two studied were in fact eclipsing systems. It is not to be expected that this record will be duplicated; indeed, Stebbins himself found that many of the systems examined later gave no evidence of light variation. Nevertheless, the field is a promising one, but the instruments demanded for its cultivation are the very sensitive modern photometers. Excellent work, however, may be done with any good photometer on the eclipsing binaries of greater light range, and we may expect a considerable addition to the number of orbits of such systems in the near future if the observations are carried out systematically. The description of the instruments and methods of observation may be found in books on variable stars like Miss Furness's "Introduction to the Study of Variable Stars" or Father Hagen's "Die Veranderliche Sterne."

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Shapley, H. The Orbits of Eighty-seven Eclipsing Binaries—A Summary. *Astrophysical Journal*, vol. *38*, p. 158, 1913.

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These papers contain references to many others by the same, and by other writers. Attention should also be called to the recent Bulletins of the Laws Observatory, University of Missouri, for Professor R. H. Baker has entered upon an extensive program of work on eclipsing binaries and the results are being published in those Bulletins.

CHAPTER VIII

THE KNOWN ORBITS OF VISUAL AND SPECTROSCOPIC BINARY STARS

The orbits of 112 visual and of 137 spectroscopic binary star systems have been computed by different astronomers,[1] by means of the methods outlined in preceding chapters. Many of these orbits, especially those of spectroscopic binaries and of short period visual systems, may be regarded as definitive, others, especially those of the very long period visual binaries, have little or no value. Every computation was undertaken with the immediate object in view of representing the observed motion and of predicting the future motion in the particular system on the assumption that the controlling force is the force of gravitation; but back of this lay the broader motive of providing additional data for the study of the greater questions of stellar motions and, particularly, of stellar evolution. In the present chapter I shall present some relations and conclusions which may be deduced, with more or less certainty, from the computed orbit elements, but their interpretation and their bearing upon the problems of the origin of the binary stars and their relations to single star systems will be left for a later chapter.

Table I gives the elements of eighty-seven of the visual binary star systems, divided into two groups, the first containing the orbits which are at least fairly good approximations, the second, the less accurate orbits. The dividing line is not a very definite one; several orbits included in either one of the two groups might with perhaps equal propriety find a place in the other. In each group the systems are arranged in the order of right ascension, and the columns, in order, give the star's name, the position for 1900.0, the magnitudes and spectral class (taken from the *Harvard Photometry* as far as

[1] Written in October, 1917. Tables I and II, containing the orbits of these stars, are placed at the end of the book for convenient reference.

possible), the orbit elements, and the name of the computer. Only one orbit is given for each system, the most recent one, unless there is a special reason for choosing an earlier one. For many systems ten or more orbits have been published. References to the earlier orbits may be found in Burnham's *General Catalogue* or in Lewis's volume on the stars of the *Mensurae Micrometricae*. The later orbits are, for the most part, published in the *Astronomische Nachrichten*, the *Astronomical Journal*, and Volume 12, of the *Publications of the Lick Observatory*. The orbits which have been computed for twenty-five other systems are omitted because the data upon which they are based are entirely inadequate, and any conclusions drawn from the results would be entitled to no weight whatever. Perhaps the best observed among the omitted systems is the well known binary *σ Coronae Borealis;* using practically the same observations, Lewis finds the revolution period to be 340 years, while Doberck's value is 1,679 years!

For reasons already given, the orbits of the spectrographic binary stars are, in general, more accurate than those of the visual binaries, and only five of the 137 are assigned to the second group of Table II. Of these, *α Persei* is excluded from the main table because there is still a reasonable doubt as to whether or not it is a binary at all[2]; *β Canis Majoris* and *ρ Leonis* because the orbits are avowedly only rough approximations, and the two long period systems, *α Orionis* and *α Scorpii*, because Lunt's[3] recent discussion shows that the data on which the orbits rest are inadequate for good solutions. It does not follow that all of the orbits retained in the main division of the table are of equal value; doubtless, several should properly be transferred to the second section for reasons similar to those given for the transfer of *α Orionis* and *α Scorpii*. A few others depend on the H and K lines of calcium only, lines which exhibit anomalies not yet perfectly understood; and one system, *φ Persei*, has recently been pronounced as still "a complete riddle." The results which follow, however, are not affected by their retention.

[2] *Lick Observatory Bulletin*, 7, 99, 1912.
[3] *Astrophysical Journal*, 44, 250, 1916.

As in the table for the visual systems, the stars are arranged in order of right ascension. The columns give, consecutively, the star's name, the position for 1900.0, the magnitude and spectrum (from the *Harvard Photometry*) the orbit elements and the computer. Here again but one orbit is given, though two or more have been computed for several systems. In a number of systems, however, the second spectrum is visible on the plates, and values of the elements of the secondary, which differ from those of the primary, if given by the computer, are entered in the line below the principal orbit. In a few instances, also, the elements of a third body in the system, or of a secondary oscillation in the orbit are added.

RELATIONS BETWEEN PERIOD AND ECCENTRICITY

Certain striking characteristics of the orbits in the two tables are recognized on the most casual inspection; for example, the eccentricity of the visual orbits is generally large, that of spectroscopic orbits generally small; the periods of the former are long—the shortest known so far being 5.7 years (δ *Equulei*)—those of the latter generally short, ranging from about five hours to less than 150 days with but few exceptions.

See, Doberck, and many other recent writers have called attention to the high average eccentricity of the visual binary star orbits, See finding in this fact strong support for his earlier theory (since abandoned by him, but not entirely by others) of the origin of the binary star systems. The average value of the eccentricity for the eight major planets in the solar system is only 0.06, and the largest, for *Mercury*, is only 0.206; whereas the average of the sixty-eight values in the first part of Table I is 0.48, agreeing very precisely with the value See found in 1896 from the orbits of forty systems. The average value of *e* for the nineteen systems in the second group is even higher, exceeding 0.61.

On the other hand, the average eccentricity of the 132 orbits of spectroscopic binaries included in the main division of Table II is only 0.205, and of the five more uncertain orbits only 0.30. Recalling the fact that the periods of the visual binaries are, on the average, much greater than those of the spectroscopic, it is natural to try to establish a relation between the two elements. Doberck long ago presented evidence tending to show that the eccentricity of the visual binaries increased with the length of the period; Campbell, Schlesinger, Ludendorff and others have shown that a similar relationship exists among the spectroscopic binaries.

Examining the data now available, we find that the relationship is established beyond question. Omitting the twelve Cepheid variable stars [4] and RR *Lyrae*, a 'cluster type' variable, because the peculiarities in these systems seem to differentiate them too much from the other spectroscopic binaries to justify their inclusion in studies of the relations between orbit elements, I have classified the remaining 119 stars in the main part of Table II according to period and eccentricity and give the results in Table III. Table IV contains a similar grouping for the sixty-eight visual binaries of the first part of Table I.

TABLE III

Periods and Eccentricities in Spectroscopic Binaries

$e \backslash P$	d 0–5	d 5–10	d 10–20	d 20–50	d 50–150	d 150+	Sums
0–.10	40	9	6	3	3	1	62
.10–.20	5	4	1	0	2	4	16
.20–.30	1	5	1	1	2	2	12
.30–.40	0	0	2	1	2	1	6
.40–.50	0	1	0	2	1	3	7
.50–.60	0	0	1	3	3	2	9
.60–.70	0	0	1	0	0	1	2
.70–.80	0	0	0	3	1	0	4
.80–.90	0	0	0	0	1	0	1
.90–1.00	0	0	0	0	0	0	0
Sums	46	19	12	13	15	14	119

[4] *β Cephei* should perhaps be classified with the Cepheid variables, for according to Guthnick, the light curve resembles that of variables of this class. The variation, however, is only 0.05 magnitude, and in its spectral class and the characteristics of its orbit it differs decidedly from the twelve Cepheids which I am discussing separately.

TABLE IV

Periods and Eccentricities in Visual Binaries

e \ P	y 0–50	y 50–100	y 100–150	y 150+	Sums
0–.10	0	0	0	0	0
.10–.20	5	1	0	1	7
.20–.30	4	0	0	0	4
.30–.40	7	4	2	1	14
.40–.50	5	6	1	2	14
.50–.60	4	4	1	5	14
.60–.70	1	1	1	1	4
.70–.80	3	3	0	1	7
.80–.90	1	1	0	1	3
.90–1.00	0	0	1	0	1
Sums	30	20	6	12	68

A more succinct summary may be made as follows:

TABLE V

Spectroscopic Binaries

Number	Average Period	Average Eccentricity
	d	
46	2.75	0.047
19	7.80	0.147
12	15.17	0.202
13	30.24	0.437
15	106.4	0.371
14	1,035.	0.328

Visual Binaries

Number	Average Period	Average Eccentricity
	y	
30	31.3	0.423
20	74.4	0.514
6	124.5	0.558
12	243.	0.529

To smooth the relationship curve let us combine lines 3 and 4, 5 and 6, and 9 and 10 of Table V; we then have:

TABLE VI

	P	e
Sp. Bin. 46	2.75 da.	0.047
19	7.80	0.147
25	23.00	0.324
29	555(=1.5 y.)	0.350
Vis. Bin. 30	31.3 yr.	0.423
20	74.4	0.514
18	170	0.539

It is of interest, however, to note in Table V the definite maximum of e in spectroscopic binary stars with periods of from twenty to fifty days, and the similar, but less marked, maximum in visual binaries with periods between 100 and 150 years. The latter may be apparent, only, since but six stars are involved. The rapid increase of e with lengthening P in the first three lines of Table VI and the relatively slow increase in the later lines is worthy of note.

The evidence from the less certain orbits may be added, for the sake of completeness; the three short period spectroscopic binary stars have an average period of 5.5 days and an average e of 0.36; while for the two long period ones the data are $P = 5.99$ yr., $e = 0.22$. Had these results been included in Table V the general order of the averages would not have been affected. This remark holds also for the nineteen uncertain orbits of visual binaries, but here the last two *numerical* values would have been changed materially, for we have:

6 stars average P 67.7 yr. average e 0.51
6 stars average P 118 yr. average e 0.68
7 stars average P 213 yr. average e 0.65

The relationship so definitely established between the length of the revolution period and the degree of ellipticity of the orbit must have a physical significance.

RELATIONS BETWEEN PERIOD AND SPECTRAL CLASS

Dr. Campbell, in his study of the spectroscopic binary stars found evidence of a relationship between the period and the spectral class; taking the spectra in the order B, A, F, G, K, and M the period increases as we pass from B toward M. Before analyzing the present data to see whether they support this conclusion, it should be said that in combining the various Sub-classes, I have followed the Harvard Observatory system, making Class B include Sub-classes O to B8, Class A, Sub-classes B9 to A3, Class F, Sub-classes A5 to F2, Class G, Sub-classes F5 to G0, Class K, Sub-classes G5 to K2, and Class M, Sub-classes K5 to Mb. This differs somewhat from the grouping adopted by Campbell; a fact which must be kept in mind if comparisons are made between his tables and those which follow.[5] It must also be noted that Table I contains a few exceptional stars that cannot at present be fairly considered in this connection. These are χ *Aurigae*, Class B1, whose period of 655 days equals more than half the sum of all the remaining forty-seven Class B periods; γ *Geminorum*, Class A, period 2,175 days, and ϵ *Ursae Majoris*, Class A, period 4.15 years (= 1,515 days), respectively four and three times the sum of the periods of the remaining thirty-six Class A stars; and ϵ *Hydrae*, Sub-class F8, with a period of 5,588 days, as compared with 3,476 days for the sum of the remaining thirteen Class G stars.

Omitting these four stars from Table II (ϵ *Hydrae* is counted with the visual binaries) and, as before, the Cepheid and cluster-type variables, and counting the one Class M star with Class K, we have the following results:

[5] Campbell also includes a number of systems whose periods are known either definitely or as 'short' or 'long', but for which no orbits have been computed.

TABLE VII

Spectroscopic Binaries

No.	Class	Av. P	Av. e
48	B	26.76 da.	0.189
36	A	13.35	0.187
10	F	32.76	0.252
13	G	267.4	0.129
8	K–M	152.9	0.196

Visual Binaries

No.	Class	Av. P	Av. e
13	A	98.9 yr.	0.529
9	F	100.6	0.512
30	G	78.7	0.478
12	K	86.0	0.432
4	M	126.7	0.402

Table VII has several features of interest. In the first place, it appears that if we divide the spectroscopic binaries into two groups by a line between Classes F and G, those of the second group have average periods decidedly longer than the periods of those in the first; but that in neither group does the period increase, on the average, with advancing spectral class. In the second place, we note that the distribution with respect to spectral class is very different in the two sets of binaries, the spectroscopic and the visual; spectra of early type predominate among the former, whereas *there is not a single Class B star among the latter*, and nearly half the number belong to Class G.

Looking over the list of spectroscopic binaries for which no orbits have as yet been computed, I find that in fifty-three cases either the approximate period or the note 'long' or 'short' is given. Classifying these according to spectrum, I find:

	B	A	F	G	K	M
Period short	13	9	6	5	0	0
Period long	1	1	0	8	7	1

Several years ago I also tabulated the spectral classes of 164 visual binaries which show rapid orbital motion and found that four were of Class B, 131 of Classes A to F (including Sub-classes F5 and F8), 28 of Classes G to K (including K5) and only one of Class M.

In connection with the facts just stated it must be kept in mind that spectroscopic systems of short period are more readily discovered than those of long period, for not only is the amplitude of the velocity curve greater, in general, in the former but the variation in the velocity becomes apparent in a much shorter time. We may expect relatively more long period systems in future discoveries among stars of all classes of spectra and hence an increase in the average values of the periods. It should also be noted that the spectra of stars of the later types, in general, show more lines, and these more sharply defined, than the spectra of the early type stars. The probable error of measure is therefore less and hence a variable radial velocity of small amplitude may be unmistakable in, say a Class G or K star, whereas one of equal amplitude may escape recognition in a star of Class B. This might account in part, for the relative numbers of the systems of long period among the different classes of spectroscopic binaries, but it obviously does not explain the very large relative number of short period binaries of Class B. As for the visual binaries, if we accept the current doctrine that the stars of Class B and the brighter stars of Class M are the most remote we might expect that some systems of these classes with periods of the order of those given in the table would fall below the resolving power of our telescopes. Allowing for the effects of these factors, we may still conclude that, taking both the average periods and the number of systems into account and also the difference in these respects between the visual and spectroscopic systems the evidence is definitely in favor of an increase of the period of binary stars with advancing spectral class.

In passing, attention may be called to the curious distribution of the eccentricities in Table VII; the definite progression, or rather retrogression, with advancing spectral class shown by the visual systems is in marked contrast with the distribution among the spectroscopic systems. It is doubtful whether any significance attaches to either.

THE DISTRIBUTION OF THE LONGITUDES OF PERIASTRON

In 1908 Mr. J. Millar Barr [6] called attention to a singular distribution of the values of ω, the longitude of periastron, in those spectroscopic binaries whose orbits are elliptic. In the thirty orbits available to him in which e was greater than 0.0, twenty-six had values of ω falling between 0° and 180° and only four between 180° and 360°. Since there is no conceivable relation between the position of the longitudes of periastron in these orbits and the position of the Sun in space (except in the case of the Cepheid variables), such a distribution is inherently improbable unless it is produced by some error of observation. This, in fact, was Barr's conclusion—"the elliptic elements, e and ω, as computed and published for the orbits under notice, are probably illusory, the 'observed radial velocities' upon which they are based being vitiated by some neglected source of systematic error."

If this conclusion were well founded, it would be a serious matter indeed, but both Ludendorff and Schlesinger have shown that the one-sided distribution of periastra noted by Barr "was nothing more than a somewhat extraordinary coincidence," for it becomes less marked as additional orbits are considered. The data in Table II tend to confirm this statement, though the inequality of distribution is not yet entirely eliminated. Omitting the Cepheid variables and three stars for which ω might be taken on either side of the dividing line (2 *Lacertae*, ω = 180°, β *Lyrae*, ω = 0°, and π *Cassiopeiae*, a system with practically equal components either of which might be regarded as the primary) we have 100 systems in which e exceeds zero. Dividing these into two groups according to the value of e, the values of ω are distributed in the four quadrants as follows:

	1st	2nd	3rd	4th
$e < 0.50$	26	21	19	16
$e \gtreqless 0.50$	9	3	3	3

[6] *Journal R. A. S. C.*, 2, 70, 1908.

The ratio of the number of ω's less than 180° to that of the ω's greater than this value is thus 47 : 35 and 12 : 6 in the two groups respectively. Doubtless, when the number of known orbits is doubled, these will both be reduced approximately to equalities.

THE MASSES OF THE BINARY STARS

A knowledge of the masses of the stars is one of the fundamental requirements for a solution of the mechanical problems of our stellar system, and this knowledge can be derived only from binary stars. For this reason the methods by which we determine the absolute and the relative masses in both visual and spectroscopic binaries, and the results which have been obtained by their application, will be presented in some detail.

Unfortunately, the orbit elements alone do not afford all the data necessary for the determination of either mass or density. The well-known harmonic law $D^3 : d^3 = P^2(M + M_1) : p^2 (m + m_1)$, will give the mass of any system in terms of the Sun's mass when the linear dimensions of the system as well as the orbit elements are known. But the semi-axis major of the visual binary stars is known only in terms of seconds of arc, and its value, so expressed, must be divided by its parallax to reduce it to linear measure[7]; and we do not know the true semi-axis major of the spectroscopic binary orbits at all, but only the function $a \sin i$. This, however, is expressed in kilometers.

The parallax of a number of visual binaries is known with a greater or less degree of certainty, and mass values for those systems may be computed, using the harmonic law in the form

$$(m + m_1) = \frac{a^3}{\pi^3 P^2} \qquad (1)$$

in which π is the parallax of the system, P the period and a the semi-axis major of its orbit, and the units of mass, length and time are respectively, the Sun's mass, the astronomical unit, and the year.

While we are unable to derive the mass of any given spectroscopic binary until we have a knowledge of the value of i,

[7] This division gives the length in astronomical units. The astronomical unit is the Earth's mean distance from the Sun, in round numbers, 149,500,000 kilometers.

the inclination, we may nevertheless estimate the *average* mass of a number of systems with approximate accuracy, by determining the probable average value of i and hence of $\sin i$. The formulae required differ for the two cases (1) when both spectra have been observed, and (2) when only one spectrum is visible. They may be derived from the well known relation

$$(m + m_1) = \frac{4\pi^2}{k^2} \cdot \frac{(a + a_1)^3}{P^2} \qquad (2)$$

in which π denotes, not the parallax, but the circumference of radius unity, k the Gaussian constant (log. 8.23558), a and a_1, the major semi-axis of the orbits of the primary and secondary respectively, and P their revolution period. Since we do not know a but only the function $a \sin i$, we must multiply both members of (2) by $\sin^3 i$, and since $a \sin i$ is expressed in kilometers, we must divide its value by that of the astronomical unit A expressed in kilometers. The numerical value of $4\pi^2/k^2A^3$ is approximately $4/10^{20}$ and we therefore have

$$(m + m_1) \sin^3 i = \frac{4}{10^{20}} \cdot \frac{(a \sin i + a_1 \sin i)^3}{P^2}. \qquad (3)$$

From equation (9) of Chapter IV,

$$a \sin i = [4.13833]\, KP \sqrt{1 - e^2}$$

hence

$$(m + m_1) \sin^3 i = [3.01642 - 10]\, (K + K_1)^3 P(1 - e^2)^{3/2} \qquad (4)$$

the numbers in square brackets being logarithms. This equation is independent of the parallax, or distance of the system.

When both spectra have been measured and the corresponding velocity curves drawn we obtain at once the *relative masses* of the two components, from the relation $m : m_1 = K_1 : K$; and we also have the equations

$$\left.\begin{aligned} m \sin^3 i &= [3.01642 - 10]\, (K + K_1)^2 K_1 P(1 - e^2)^{3/2} \\ m_1 \sin^3 i &= [3.01642 - 10]\, (K + K_1)^2 K\, P(1 - e^2)^{3/2} \end{aligned}\right\} \qquad (5)$$

from which to compute the masses of the components separately.

When only one spectrum is visible we must apply a somewhat different formula, namely,

$$\frac{m_1^3}{(m+m_1)^2} \sin^3 i = \frac{4}{10^{20}} \frac{(a \sin i)^3}{P^2} \qquad (6)$$

in which $a \sin i$ and m refer to the component whose spectrum is given. We may write this in a form similar to equation (4) thus:

$$\frac{m_1^3 \sin^3 i}{(m + m_1)^2} = [3.01642 - 10] K^3 P \, (1 - e^2)^{3/2}. \qquad (7)$$

In applying equations (4) and (7) it is necessary to assume a value for $\sin^3 i$ and the question of obtaining such a value has next to be considered. "It can be shown for an indefinitely great number of binary systems whose orbital planes are distributed at random, that the average inclination would be 57.3°, in accordance with the formula

$$i_0 = \frac{2}{\pi} \int_0^{\pi/2} \int_0^{\pi/2} i \sin i \, d i \, d \phi = 1.$$

The average value of $\sin^3 i$, however, would not be $\sin^3 57.3°$ (= 0.65), but approximately 0.59 in accordance with the formula

$$\sin^3 i_0 = \frac{2}{\pi} \int_0^{\pi/2} \int_0^{\pi/2} \sin^4 i \, d i \, d \phi = {}^3/_{16} \pi = 0.59."$$

Campbell, whom we have just quoted, and Schlesinger, who, from a slightly different formula obtains the same value for $\sin^3 i_0$, point out that while this mean value holds for orbits in general it would not be permissible to use it for the spectroscopic binary stars whose orbits have so far been computed. For, to quote again from Campbell, "there is the practical consideration that binary systems whose orbital planes have large inclinations are more readily discoverable than those whose inclinations are small . . . Under ordinary circumstances, and when dealing with a considerable number of orbits, a compromise value of $\sin^3 i = 0.65$ might in fairness be adopted." For eighteen systems which he actually considers he adopts the higher value 0.75 because six of them are eclipsing binaries, with inclinations quite certainly between 60° and 90°.

Schlesinger, assuming "that the chance of discovery is proportionate to $\sin i$", obtains $\sin^3 i = 0.68$ for a mean value. We may then adopt, for convenience in computation, $\sin^3 i = 0.667 = 2/3$, since comparatively few eclipsing binaries are among the number under discussion. Considering first the spectroscopic binaries in which two spectra have been observed and for which either the values of $m \sin^3 i$ and $m_1 \sin^3 i$ or the ratios m_1/m are given by the computer of the orbit, we have the data in Table VIII.

TABLE VIII

Relative Masses of Spectroscopic Binaries

		$m \sin^3 i$	$m_1 \sin^3 i$	m	m_1	m^1/m
Boss 6142	Bp	18.5	12.7	27.8	19.0	0.69
o Persei	B1	5.42	3.79	8.1	5.7	0.70
η Orionis	B1	11.2	10.6	16.8	15.9	0.95
β Scorpii	B1	13.0	8.3	19.5	12.4	0.64
α Virginis	B2	9.6	5.8	14.4	8.7	0.60
β Lyrae	B2p	(6.8	16.6)	(10.2	24.9)	(2.46)
4 Androm.	B3	1.50	1.10	2.2	1.6	0.73
λ Tauri	B3					0.40
u Herc.	B3	7.5	2.9	(11.2)	(4.4)	0.39
57 Cygni	B3	1.79	1.67	2.7	2.5	0.93
2 Lacertae	B5					0.81
α Aquilae	B8	5.3	4.4	8.0	6.6	0.83
υ Erid.	B9	5.58	5.48	8.4	8.2	0.98
ψ Orionis	A	5.53	4.19	8.3	6.3	0.76
136 Tauri	A					0.69
β Aurigae	Ap	2.21	2.17	3.3+	3.3−	0.98
40 Aurigae	A	1.35	1.11	2.0	1.7	0.82
Boss 2184	A	1.48	1.27	2.2	1.9	0.86
ω Urs. Maj.	A					0.17
d_2 Virg.	A					0.56
ζ_1 Urs. Maj.	Ap	1.70	1.62	2.6	2.4	0.95
ε Herc.	A	1.6	1.0	2.4	1.5	0.62
108 Herc.	A	0.94	0.89	1.4	1.3	0.95
50 Drac.	A	0.90	0.82	1.4	1.2	0.91
2 Sagittae	A	0.91	0.65	1.4	1.0	0.72
θ Aquilae	A	0.52	0.38	0.8	0.6	0.73
b Persei	A2					0.28
π Cass.	A5	1.32+	1.33−	2.0	2.0	1.003
Boss 4423	F					0.96
o Leonis	F5p	1.30	1.12	2.0	1.7	0.86
d Boötis	F5	1.36	1.29	2.0	1.9	0.95
α Aurigae	G	1.19	0.94	1.8	1.4	0.79

Columns three and four of the table give the minimum values of the masses, for it is clear that m is a minimum when $\sin^3 i$ is placed equal to unity. It appears from these columns that in only one system, θ *Aquilae*, may the separate components be regarded as probably less massive than the Sun.

The system of β *Lyrae* is in many respects a peculiar one and there are exceptional difficulties in interpreting its spectrum. It appears from the last column of the table that it is the only one in which the fainter star is definitely the more massive. Omitting it, the average mass ratio, m^1/m, in the remaining thirty-one systems is 0.748. Those who have examined the spectra of large numbers of stars have also noted, as Schlesinger says, that there appears to be a close correspondence between relative mass and relative brightness of the components; when the two spectra are almost equally conspicuous, the two masses are also about equal, but when one spectrum is barely discernible the corresponding mass is also relatively small. Schlesinger adds "we may infer that in those binaries in which the fainter component does not show at all, the mass of the brighter star is all the more preponderant." It must be emphasized that the numbers set down in the two columns m and m_1 *have no meaning so far as any particular system is concerned* [8]; the value 0.667 is the *mean* value for $\sin^3 i$, but in a particular system it may have any value from nearly zero to unity. It is also apparent that the means of the two columns cannot properly be taken as the average masses in spectroscopic binary systems, for the individual results show a definite relation to the spectral class. Omitting β *Lyrae*, we have

	m	m_1	m_1/m
B to B8 (9 stars)	12.3	8.5	0.69
B9 to A5 (12 stars)	3.0	2.4	0.80
F5 to G (3 stars)	1.9	1.7	0.89

[8] In the eclipsing variable u *Herculis* there is reason to think that i is approximately 90°, and that the masses in columns three and four are the true masses.

And from the last column

	m_1/m
B to B8 (11 stars)	0.70
B9 to A5 (16 stars)	0.75
F to G (4 stars)	0.89

From this summary it appears that the systems of very early type are decidedly more massive than the others, and that all of the twenty-four systems for which m and m_1 are given are more massive than the Sun. There also appears to be evidence of an unexpected progression in the ratio of the masses of the two components, the secondary in systems of earlier type being less massive relatively to its primary, than the secondary in those of later type. It is somewhat remarkable that twenty-six of the thirty-one stars in the table have spectra of early type (B to A2) and that only one has a spectrum as late as Class G.

These conclusions cannot legitimately be extended to all spectroscopic binary star systems, for the systems under discussion are *selected* stars in the sense that it is only in those systems in which the two spectra are well separated—and the values of K therefore large—that the spectrum of the fainter component is visible. The sum $(K + K_1)$ enters by its cube as a factor in equation (4), and the mass, therefore, in general, increases very rapidly with K. In fact, the average K for the primary stars in these thirty-two systems (β *Lyrae* included) is almost precisely double that of the eighty-one single spectrum systems (excluding the Cepheid variables) for which the computers publish this element.

The value of the function $\frac{m_1^3 \sin^3 i}{(m+m_1)^2}$ is frequently omitted by the computer of orbits for it gives very little definite information. Equation (7) affords a ready means of computing the function for any system, but I have not considered it necessary to carry out the computation, for a glance at the numbers recorded in Table II shows at once that no conclusions could be based upon any means that might be taken. For *Polaris* this function equals 0.00,001 ⊙ [9], for 29 *Canis Majoris*, 4.58 ⊙, a range of 1 to 460,000; while for the exceptional system *β Lyrae*, it is 8.4 ⊙. It is apparent, however, that in the single spectrum binaries the secondary is considerably less massive than its primary in nearly all cases unless we are willing to adopt improbably small values for $(m + m_1)$ and for $\sin^3 i$.

The present evidence may therefore be summed up in the general statement that in the spectroscopic binaries with known orbits, whether one spectrum or both spectra have been observed, the brighter star is, with very few exceptions, the more massive; binary stars of Class B have masses decidedly greater than the stars of other classes, and binary systems of all classes have masses greater than that of the Sun. There is nothing novel in these conclusions; they simply confirm the results obtained by earlier investigators. Ludendorff, for example, showed quite conclusively that among the systems available for study in 1911 those of Class B were, on the average, about three times as massive as those of Classes A to K.

Passing to the visual binaries, I have computed the masses of those systems for which the orbit elements and the parallaxes are known with sufficient accuracy to give the results significance. An earlier computation, in which I used every system for which published elements and parallaxes were available regardless of their probable errors, had given results which varied through a very wide range; but it was obvious that many of these were worthless either because the orbit elements were unreliable or because the parallax was too uncertain. Equation (1) shows that the parallax is the most important factor in this case, for it enters by its cube, while the elements a and P, which in a general way vary in the same sense, to a certain degree offset each other. Thus in the system of *η Cassiopeiae*, an orbit with $P = 345.6$ years, $a = 10.10''$ would give very nearly the same mass as the adopted orbit; a change of only $0.02''$ or $0.03''$ in the value of π, on the other hand, would change the mass by thirty per cent.

[9] Symbol for Sun.

TABLE IX

Masses of Visual Binary Stars

Star	Mag.	Sp.	P_y	a	π	$(m+m_1)$
Sirius	−1.6, 9.0	A, A	49.32	7.55″	+0.376″	3.3☉[10]
o² Erid.	9.4, 10.8	A0	180.03	4.79	+0.174	0.6
δ Equulei	5.3, 5.4	F5	5.70	0.27	+0.067	2.0
Procyon	0.5, 13.5	F5	39.0	4.05	+0.31	1.5
ε Hydrae	3.7, 5.2	F8	15.3	0.23	+0.025	3.3
η Cass.	3.6, 7.9	F8	508.(?)	12.21	+0.201	0.9
85 Pegasi	5.8, 11.0	G	26.3	0.81	+0.067	2.6
ζ Herc.	3.0, 6.5	G	34.46	1.35	+0.107	1.7
η Cor. Bor.	5.6, 6.1	G	41.56	0.89	+0.06	1.9
ξ Urs. Maj.	4.4, 4.9	G0	59.81	2.51	+0.179	0.8
α Cent.	0.3, 1.7	G, K5	78.83	17.65	+0.759	2.0
70 Oph.	4.1, 6.1	K	87.86	4.56	+0.168	2.6
Krueg. 60	9.3, 10.8	Mb	54.9	2.86	+0.258	0.45
μ¹ Herc.	10.0, 10.5	Mb	43.23	1.30	+0.106	1.0
Mean mass, 14 systems						1.76☉

Table IX gives the results of my computation, the values of π which were adopted being entered in the sixth column. With two exceptions they were derived from heliometer measures or the measures of photographic plates. The two exceptions are the parallaxes for *δ Equulei* and *ε Hydrae*. The former was determined by Hussey on the basis of his orbit and the spectrographic measures of the relative radial velocities of the two components at the time of perihelion. The possibility of determining parallax by this method was pointed out by See more than twenty years ago, but in very few of the visual binary systems is it feasible to measure the *relative* velocities of the two components. The only other determination of parallax by this method known to me is Wright's, for the system of *α Centauri*. His value, 0.73″, agrees closely with that obtained by the best heliometer observations. The computation is readily made by means of the following formulae, adapted by Wright from Lehmann-Filhés's work [11]:

[10] Symbol for Sun.

[11] I have made slight changes in Wright's notation, which is given in *Lick Observatory Bulletin 3*, 1904.

Let

R = the astronomical unit, expressed in kilometers.

a = the semi-major axis of the binary, expressed in kilometers, and a'', the same element expressed in seconds of arc.

n = the mean angular motion of the star, in the visual orbit, in circular measure per second of time.

ΔV = the observed difference in the radial velocity of the two components.

Then

$$\left.\begin{aligned} n &= \frac{2\pi}{86400 \times 365.26 \times P} \\ a &= \frac{\Delta V \sqrt{1 - e^2}}{n \sin i \,[e \cos \omega + \cos(v + \omega)]} \\ \pi'' &= \frac{a''}{a} \cdot R \end{aligned}\right\} \qquad (8)$$

The system of ϵ *Hydrae* is unique in that the measures of the variable radial velocity of the bright component permit us to determine the orbit elements independently of the micrometric measures. When I computed the elements in 1912, the spectrographic observations covered somewhat less than one revolution and the period was therefore assumed from the micrometric measures; the elements e, T and ω were found to be identical in the two systems; the value of i in the visual system permitted the separate determination of a in the spectroscopic system, and this, in turn, permitted the definition of a in the visual orbit in terms of kilometers.[12] The result, 1,359,000,000 kilometers, combined with the value of the astronomical unit in kilometers, at once gave the parallax 0.025″ and hence the mass 3.33 ⊙.

The mean value of the mass of the fourteen systems in Table IV is 1.76 times the mass of the Sun. For the seven systems which he considers most reliable, Eddington [13] obtains a mean value 1.66, and Innes,[14] from eight systems, a mean

[12] It is to be remembered that in spectroscopic binaries with only one visible spectrum a is always the mean distance of the bright star from the center of gravity of the system, while in the visual orbit, a is the mean distance between the two components.

[13] Eddington, "Stellar Movements and the Structure of the Universe," p. 22.

[14] Innes, "The Masses of the Visual Binary Stars," *South African Journal of Science* for June, 1916.

value of 1.92. I may add the testimony from the report given by Miss Hannah B. Steele [15] at the nineteenth meeting of the American Astronomical Society, on the parallaxes of twenty visual binaries with known orbits measured at the Sproule Observatory. Of the sixteen positive parallax values, four had probable errors varying from one-half to four times the numerical value of the parallax. Rejecting these, Miss Steele's masses for the remaining twelve pairs range from 0.26 ⊙ to 6.25 ⊙ with an average value of 1.7 ⊙. Admitting the meagerness and the uncertainties of the data, we may still make the general statement that the visual binary systems for which we have the best orbits and parallaxes are, on the average, about twice as massive as the Sun.

The danger of drawing general conclusions from this result arises not only from the fact that the number of stars upon which it is based is so small, but even more from the fact that they are of necessity selected stars, those relatively close to us. To make this clear, let us assume the mass $(m + m_1)$ equal to twice the Sun's mass, and then use equation (1) to construct a table giving the values of a with arguments P and π,

TABLE X

The Semi-axis Major in a Binary Star System with Given Periods and Parallax

$P \diagdown \pi$	0.01″	0.005″
5y	0.04−″	0.02−″
10	0.06−	0.03−
20	0.09+	0.05−
40	0.15−	0.07+
60	0.19+	0.10−
120	0.31−	0.15+
250	0.50	0.25
700	0.99+	0.50−
2,000	2.00	1.00

Looking back to Table I, it will be found that in every single instance the mean distance a is greater, generally much greater,

[15] Miss Steele, *Popular Astronomy*, February, 1917, p. 107.

than the mean distance in the column under $\pi = 0.01''$ in Table X for the corresponding period. In other words, the systems listed in Table I *either have more than twice the mass of the Sun or their parallax exceeds 0.01''*. Now every increase in our knowledge of stellar distance makes it more certain that the average parallax of the naked eye stars is only of the order of 0.01'' and that the average becomes progressively smaller as we pass from one magnitude to the next fainter one among the telescopic stars, the best determinations for the stars of magnitudes 7, 8 and 9 being respectively about 0.009'', 0.007'' and 0.005''+. On the other hand, in the systems for which we know the parallax, the average mass is less than twice the mass of the Sun. We may therefore say that the visual binaries whose periods are known are among our nearer stellar neighbors. This does not hold true for the spectroscopic binaries because the discovery of variable radial velocity does not depend upon the distance of the star.

At the Ottawa meeting of the American Astronomical Society (1911), Russell presented the results to which he had been led by employing statistical methods in a study of the relations between the mass, density, and surface brightness of visual binary stars. Hertzsprung had earlier shown the existence of a group of stars which are entirely above the average in luminosity and probably in mass and had called them 'giants' to distinguish them from the 'dwarfs'; all Class B stars and some stars in every one of the other spectral classes are placed among the giants. Dividing 160 giant binary stars into four groups according to spectrum, Russell determined the mean mass of a system in each group, the values ranging from 7 to 13 times that of the Sun, while the mean luminosities ranged from 130 to 195 times that of the Sun, an indication that these stars have great volumes and correspondingly low densities. Similarly, for the mean masses of the systems in five groups of 'dwarf' binaries (189 systems in all), the values were found to range from 5.4 ⊙ for the Class A group to 0.4 ⊙ for the Class M group, the mean luminosity decreasing in the same direction from 25 to 0.02 times that of the Sun. Making every allowance for the uncertainties introduced by errors in the data and by estimating the average values of unknown functions on principles of probability, the results of such determinations may still be regarded as corroborative evidence which increase our confidence that the masses found by direct methods, in the few systems where such methods are applicable, are fairly representative of those in binary systems in general.

Hertzsprung has shown that we may use equation (1) to determine a minimum value for the parallax of visual double star systems in which orbital motion has been observed but for which the observed arc is too short to permit the computation of orbit elements.

Let

$$V_1 = \frac{2\pi a}{P}$$

be the orbital velocity in a circular orbit; then from equation (1) we have

$$\frac{a}{p^3} V_1^2 = 4\pi^2 (m + m_1)$$

where p is the parallax of the star. Now the velocity V in a parabolic orbit equals $V_1\sqrt{2}$, hence, using R for the radius vector instead of a, we have

$$\frac{R}{p^3} V^2 = 8\pi^2 (m + m_1).$$

But in an elliptic orbit, such as we assume for a double star, the orbital velocity must be less than the parabolic velocity, and therefore

$$p^3 > \frac{RV^2}{8\pi^2(m + m_1)}$$

and since projection can only shorten the radius vector and diminish the apparent orbital velocity we must have, *a fortiori*,

$$p^3 > \frac{rv^2}{8\pi^2(m + m_1)} \qquad (9)$$

where r and v are the projected values of R and V. The right hand member of (9) is therefore the expression for the minimum possible parallax, and when an assumption is made as to the mass, all terms in it are known, for r is given by the observed distance ρ and v^2 by the observed angular velocity $\frac{d\theta}{dt}$ · Hertzsprung assumes $(m + m_1)$ = the Sun's mass = 1 and writes for the minimum hypothetical parallax

$$p^3_{h,\ \min} = \frac{r\,v^2}{8\pi^2}. \qquad (10)$$

Comparison with stars whose parallaxes are known leads him to conclude that the ratio $p : p_{h,\ \min}$ does not vary greatly and that, in the mean

$$\log \frac{p}{p_{h,\ \min}} = +0.27 \pm 0.14 \qquad (11)$$

or, in words, the true parallax of a double star system is approximately double the minimum hypothetical parallax, the probable error being about one-third of its value. From (9) and (10) we obtain

$$(m + m_1) > \frac{p^3_{h,\ \min}}{p^3}$$

as the expression for the minimum mass in a system of known parallax. If we accept the relation expressed by (11) it follows that the mass of an average double star system exceeds one-eighth the mass of the Sun. This may be regarded as another bit of evidence favoring the conclusion that the stars in binary system are of the same order of mass as the Sun.

The orbit elements of a visual binary give us no direct information as to the position of the center of gravity of the system, nor as to the relative masses of the two components; but under favorable conditions this information may be acquired from measures connecting one of the components with one or more independent stars. When such measures, covering a sufficient time interval, are available for a system in which the angular separation is fairly large the relative masses can be determined in a very simple manner.[16]

[16] See *Astrophysical Journal*, 32, 363, 1910.

Let AB be the binary system, C an independent star, and let ρ, θ and ρ', θ', respectively, be the distance and position angle of C referred to A and of B referred to A. Then the apparent rectangular coordinates of C and B referred to axes drawn from A as origin in position angles θ_0 and $(90° + \theta_0)$ will be

$$x = \rho\cos(\theta - \theta_0) \qquad x' = \rho'\cos(\theta' - \theta_0)$$
$$y = \rho\sin(\theta - \theta_0) \qquad y' = \rho'\sin(\theta - \theta_0)$$

Now if we let K equal the mass ratio $\frac{B}{A+B}$, the coordinates of the center of gravity of AB will be Kx', Ky', and since the motion of C with respect to this point must be uniform we have

$$x = a + b(t - t_0) + Kx'; \; y = a' + b'(t - t_0) + Ky' \qquad (12)$$

t_0 being any convenient epoch.

Each set of simultaneous observations of AB and AC furnishes an equation of condition in x and one in y for the determination of the five constants a, b, a', b', K. No knowledge of the period or other elements of the binary system is involved, the accuracy of the determination of K depending entirely upon the amount of departure from uniformity of motion of B relatively to A. In *Lick Observatory Bulletin* No. 208 I have published a list of systems specially suited to the application of this method and have urged the desirability of measuring them systematically.

Van Biesbroeck [17] has recently proposed a method equally simple by which the mass ratio in visual binary systems may be determined from measures of photographs taken with long-focus telescopes, and has added a list of stars to which it may be applied with prospects of good results within comparatively few years. Up to the present time, however, our information of the relative masses in visual systems has been derived almost entirely from meridian circle observations of the absolute positions of one component or of both components combined with the orbit elements.

The most reliable values are those deduced by the late Lewis Boss and published in his *Preliminary General Catalogue of*

[17] *Astronomical Journal*, *29*, 173, 1916.

Stars for 1900.0. Adding a few others that are fairly reliable, we have the data in Table XI.

TABLE XI

Star	m'/m	Computer
η Cassiopeiae	0.76	Boss
Sirius	0.29	Boss
Procyon	0.33	Boss
ζ Cancri	1.	Seeliger
ε Hydrae AB	0.9	Seeliger
ξ Urs. Maj.	1.0	Boss
γ Virginis	1.0	Boss
α Centauri	0.85	Boss
ξ Boötis	0.87	Boss
ζ Herculis	0.43	Boss
70 Ophiuchi	0.82	Boss
Krueger 60	0.56 / 0.35	Russell, 2 solutions
85 Pegasi	1.0	Boss

The testimony of this table is in harmony with that afforded by the spectroscopic binary stars, namely, that the brighter star of the system is generally the more massive; but it is only fair to add that other computers, notably the Greenwich observers, obtain results that for some of these stars differ widely from Boss's; also that Boss himself, in the case of 85 *Pegasi*, obtained a value 1.8, but considered the uncertainties to be so great that he was justified in adopting 1.0.

DENSITIES OF THE BINARY STARS

The densities of the stars in eclipsing binary systems of known orbits may be computed, as has been shown in the preceding chapter, if the ratio of the masses of the two components is also known, and Shapley's extensive investigations indicate that the average density is small. Assuming the disks darkened to zero at the edge, he finds the Class A stars to be about 1/14, the Class B stars 1/11, and the Class

F, 1/3 as dense as the Sun; while the few stars of Class G for which orbits are known exhibit so great a range in density that average values would have no meaning. Shapley has given for the upper limit of the mean density in an eclipsing system, the simple formula [18]

$$d_{\odot} \leq \frac{0.054}{P^2 \sin^3 \frac{2\pi t}{P}}$$

in which P, the period and t, the semi-duration of the eclipse, are expressed in days and $d_{\odot}$ in terms of the Sun's density. Applying it to five systems of spectral Class F8 to G5, he obtains upper limits ranging from 0.02 ⊙ to 0.00005 ⊙. It is to be remembered that Shapley, throughout, assumes the equality of the masses of the two components.

We cannot proceed so simply in the case of the visual binaries and those spectroscopic binaries which are not also eclipsing variables because the orbit data do not include any information as to the diameter of the disks of the component stars. When, however, in addition to the orbit elements, a and P, we know the surface brightness (which is a function of the absolute temperature and thus of the spectral class), and the ratio of the masses of the components, the density may be computed. Such a computation has recently been carried out by E. Öpik[19] for the principal component (both components in the system *α Centauri*) of each of thirty-nine visual binaries. The results obtained are necessarily only rough approximations because all of the data are more or less seriously affected by errors of observation. Nevertheless, they are of decided interest, for it is probable that in their densities as in other physical conditions, the visual binaries are more nearly representative of the average stars (excluding the 'giant' stars) than the eclipsing binary systems. The extreme range in Öpik's tables is from 0.012 to 5.9 the Sun's density, twenty-six values fall between 0.16 and 1.45, and the average of the forty is 0.39. A relation between spectral class and density is indicated, the stars of Class A0—A5 being the densest

[18] *Astrophysical Journal* 42, 271, 1915.
[19] *Astrophysical Journal* 44, 292, 1916.

(0.65 ☉) and those of Class K—K5 the least dense (0.072 ☉); Class B is not represented.

I have already mentioned Russell's statistical studies which lead him to assign extremely small densities and high luminosities to some of the giant stars and high densities and low luminosities to some of the dwarfs. Weighing the evidence given by the visual as well as by the spectroscopic binaries, it appears that while we may regard our Sun as a fairly typical star in point of mass, it is hardly possible to use the expression 'typical star' when we speak of density or luminosity. Consider *Sirius*, for example; the bright star is only three and one-half times as massive as its companion and about two and one-half times as massive as the Sun, but it is more than 11,000 times as luminous as the former and fully thirty times as luminous as the latter.

THE PARALLELISM OF THE ORBIT PLANES OF THE VISUAL BINARY STARS

A number of investigations have been made to ascertain whether the orbit planes of the visual binary stars exhibit a random distribution or whether there is a tendency to parallelism to a particular plane, as for example, the plane of the Milky Way. These investigations have, as a rule, been based upon the systems whose orbits are known although the fact that the orbit elements do not define the plane uniquely presents a serious difficulty. In only three or four cases has the indetermination in the sign of i, the inclination, been removed by spectrographic measures, and the true pole of the orbit thus distinguished from the 'spurious' pole. Miss Everett,[20] See[21] and Doberck[22] reached negative conclusions; their researches gave no definite evidence that the poles of the orbits favored any special region of the celestial sphere. Lewis and Turner[23] concluded that the evidence indicated, though somewhat doubtfully, a tendency of the poles to group themselves

[20] Alice Everett, *Monthly Notices, R. A. S.*, *56*, 462, 1896.

[21] T. J. J. See, *Evolution of the Stellar Systems*, *1*, 247, 1896.

[22] W. Doberck, *Astronomische Nachrichten*, *147*, 251, 1898, and *Astronomische Nachrichten*, *179*, 199, 1908.

[23] T. Lewis and H. H. Turner, *Monthly Notices*, *67*, 498, 1907.

on or near the Milky Way. Bohlin[24], on the contrary, reached the conclusion that the poles may be divided into two groups, one favoring a point near the pole of the galaxy, the other a point near the pole of the ecliptic and the apex of the Sun's way. Professor J. M. Poor[25] has recently attacked the problem by a different method based upon the thesis that "were the orbit-planes of binary stars parallel, then because the apparent orbits of those situated on the great circle parallel to their orbit-planes would be straight lines, while at the poles of this great circle the apparent orbits would be ellipses, the parallelism would show itself in a statistical study as a variation in correlation between position angle and distance of doubles in different parts of the sky." Using the data given in Burnham's *General Catalogue* and the later lists of double stars (to 1913) he reached the conclusion that a 'preferential pole' near the vertex of preferential motions of the stars was indicated.

In view of these divergent results we may consider the question as one of the many which still remain to be answered.

THE CEPHEID VARIABLES[26]

The Cepheid variables entered in Table II have been omitted from the later tables because, considered as binary systems, they seem to belong in a class by themselves. Every variable of this type which has been investigated with the spectrograph has shown a variable radial velocity and the period indicated by the velocity curve has in every instance been equal to the period of light variation. On the one hand this has been regarded as sufficient proof that these variables are binary systems; on the other, that the light variation is in some manner caused by the interaction of the two components. But many difficulties are met in attempts to construct a theory for the nature and cause of the variation, and no theory that is satisfactory in all respects has as yet been formulated. Eclipse phenomena certainly do not enter, at least in the pro-

[24] K. Bohlin, *Astronomische Nachrichten*, *176*, 197, 1907.

[25] J. M. Poor, *Astronomical Journal*, *28*, 145, 1914.

[26] I include under this head the Geminid variables also, for the two classes have no real dividing line.

duction of the principal minimum, for the epoch of this minimum does not coincide even approximately with an epoch of zero relative radial velocities of the components. On the contrary, it has been shown by Albrecht and subsequent investigators of orbits of these stars that the epoch of maximum light agrees closely with the time of maximum velocity of approach of the bright star, and the epoch of minimum light nearly as closely with the time of maximum velocity of recession; that is, as Campbell puts it, the epochs of maximum and minimum light are functions of the observer's position in space. Certain observed irregularities in the light curves of many of the Cepheids and in the velocity curves of several for which orbits have been computed, the changes in color and spectral type which the Mount Wilson observers have shown to accompany the light variation, and the independent determinations by Hertzsprung and Russell that these stars are of great absolute brightness and probably of very great volume are additional elements of difficulty. In fact, some astronomers have raised the question whether the observed line-displacements in the spectra of these stars really indicate orbital motion in a binary system or whether they may not have their origin in physical conditions prevailing in the atmospheres of single stars. The majority of astronomers, however, still hold to the opinion that they are binary systems.

Adopting the latter view, let us examine the characteristics of the computed orbits of the twelve Cepheid variables in Table II and of RR *Lyrae*, a 'cluster-type' variable. This star is properly included with the Cepheids for it is becoming apparent that the chief distinction between the two classes arises from the very short periods of the cluster variables.[27] The star β *Cephei* is probably a Cepheid variable, but, as already noted, it differs entirely in its spectrum and in most of the characteristics of its orbit from the other systems of this class, and closely resembles in these particulars the short

[27] RR *Lyrae*, however, has an annual proper motion of $0.25'' \pm$, which is so much greater than that of the average Cepheid that, on the usual assumptions as to the relations between parallax and proper motion, its luminosity is of the order of the Sun's, while the Cepheids, according to Hertzsprung, exceed the Sun in absolute brightness by about seven magnitudes.

period binaries which show no light variation. I have therefore included it with the latter and omit it here.

TABLE XII

Cepheid Variable Stars in order of Revolution Period

Star	Sp.	P	e	ω	$a \sin i$	$\frac{m_1^3 \sin^3 i}{(m+m_1)^2}$
RR Lyrae	F	0.567d	0.271	96.85°	166,000km	0.0006
SZ Tauri	F8	3.148	0.24	76.66	460,000	0.0004
RT Aurigae	G	3.728	0.368	92.016	856,000	0.0018
SU Cygni	F5	3.844	0.21	(345.8)	1,350,000	0.0058
Polaris	F8	3.968	0.13	80.0	164,000	0.00001
T Vulpec	F	4.436	0.440	104.03	966,000	0.0018
δ Cephei	G	5.366	0.484	85.385	1,270,000	0.0030
Y Sagit.	G	5.773	0.16	32.0	1,485,000	0.0040
X Sagit.	F8	7.012	0.40	93.65	1,334,000	0.0016
η Aquilae	G	7.176	0.489	68.91	1,773,000	0.0043
W Sagit.	F5	7.595	0.320	70.0	1,930,000	0.0050
ζ Gemin.	G	10.154	0.22	333.	1,798,000	0.0023
Y Oph.	G	17.121	0.163	201.7	1,790,000	0.0008

The spectra of all thirteen stars fall within the limits F to G of the Harvard scale. In general, variables of this type are almost wholly unknown among stars of spectral Classes B, A, M or N. Tabulating the data for the fifty-three known in 1910, Campbell found one of Class A, forty of Classes F to K5, and twelve of unknown class.

From the column e in the table it appears that the relations between eccentricity and period which have been established for the other binaries do not hold good for these systems, the average eccentricity of the thirteen (average period 6.15 days) being 0.300, more than double that given in Table V for the systems with average period 7.8 days. Nor is the increase of eccentricity with period very definite, though if we divide the stars into three groups according to period (0 to 5, 5 to 10, and 10 to 20 days) we find:

6 stars Av. $P =$ 3.28 days   Av. $e = 0.276$
5 stars Av. $P =$ 6.58 days   Av. $e = 0.371$
2 stars Av. $P =$ 13.64 days   Av. $e = 0.192$

The relation between period and spectral class is necessarily quite indeterminate; the last seven stars in the table, however, on the average, belong to a somewhat later spectral class than the first six.

The values of $a \sin i$ are all less than 2,000,000 kilometers, "which is evidence," in Campbell's words, "that the primaries revolve in orbits whose dimensions may be described as minute." All of the values of $\frac{m_1^3 \sin^3 i}{(m + m_1)^2}$ are also remarkably small, the largest being less than 0.006 ⊙ and the average for the thirteen only 0.0024 ⊙, which is far below the average for the other spectroscopic binaries. Three factors enter into this quantity, the orbital inclination, the mass of the system, and the ratio of the masses of the components. That the resulting function should in every case be so small argues again for the similarity of physical conditions in all of these systems. If the light variation is due either directly or indirectly to the binary character of these systems it is highly improbable that the inclinations of the orbit planes are small, but if we adopt for each of these systems the value $\sin^3 i = 0.667$ which was used in Table VIII, the function $\frac{m_1^3 \sin^3 i}{(m + m_1)^2}$ for the thirteen has an average value of only 0.0036 ⊙ and we must conclude either that the systems are much less massive than the Sun or that the ratio of the mass of the secondary to that of the bright star is very small. Ludendorff, among others, has shown that the former alternative is entirely improbable; but if the bright star is to equal the Sun in mass, it must, to produce the average value just given, be six times as massive as its secondary, a ratio far greater than that established in Table VIII.

Finally, we may note that in ten of the systems the value of ω falls not only in the first or second quadrant, but between the narrower limits 32° and 104°, the average being 80°. According to Curtiss this grouping, first pointed out by Ludendorff, is to be expected; for the observed tendency to synchronism between the epochs of light maximum and of maximum velocity of approach, and between the epochs of light minimum and of maximum velocity of recession, combined with the well-known tendency toward rapid increase and slow decrease of light in variables of this type must create a tendency toward the location of periastron on the descending branch of the velocity curve. Curtiss questions the correctness of the published value of ω for SU *Cygni* and gives plausible reasons for the discrepant values of this element in the last two systems of the table.

MULTIPLE STARS

In 1781, Herschel noted that the brighter star of the 5″ pair, ζ *Cancri*, discovered by Tobias Mayer in 1756, was itself a double star with an angular distance of only 1″ between its nearly equal components. In the years that have followed, a large number of such triple systems, and not a few that are quadruple, or multiple, have become known. During the Lick Observatory double star survey, for example, I catalogued at least 150 such systems previously unknown, and Professor Hussey's work yielded a proportionate number. The triple was formed, in more than half of these cases, by the discovery of a close companion to one of the components of a wider pair previously catalogued by other observers, and in some cases there is no question but that the closer pair had been overlooked at the earlier date because it was below the resolving power of the telescope.

The spectrograph has also revealed many triple and multiple systems; sometimes, as in 13 *Ceti* or κ *Pegasi*, by showing that one component of a visual binary is itself a binary too close to be seen as such with the telescope; again, as in *Polaris*, by showing that the short period spectroscopic binary revolves in a larger orbit with a third invisible star. Though I have made no complete count, I think it a fair estimate that at least four or five per cent. of the visual binaries are triple or quadruple systems. It seems to be a general rule that the distance between the components of the close pair in such systems whether visual or spectroscopic is small in comparison to that which separates the pair from the third star, and an argument has been based upon this fact to support a particular theory of the origin of binary systems, as we shall see in a later chapter. However, there are exceptions to the rule. Thus we have in Hu 66, BC = 0.34″, A and BC (= OΣ 351) = 0.65″; in A 1079, AB = 0.23″, AB and C = 0.48″; in A 2286, AB = 0.34″, AB and C = 0.94″; in A 1813, AB = 0.20″, AB and C = 0.70″; and in Hu 91, BC = 0.15″, AB (= OΣ 476) = 0.54″. The system of *Castor* affords an extreme example of the contrasting distances between the close and wide pairs in a quadruple star; each component of the visual pair is a spectroscopic binary, the revolution periods being respectively three and nine days while the period of the orbit described by these two pairs is certainly greater than 300 years! The motion of the third star with respect to the closer pair in a triple visual system has in no instance been observed over an arc long enough to permit the computation of a reliable orbit.

The results for the mass and density of the binary stars, and for the relations between the orbital elements which have been set forth on the preceding pages rest upon comparatively small numbers of stars, and these are, to a certain degree at least, selected stars, as has been remarked. When the number of reliable orbits has been doubled, as, from present indications, it will be within two or three decades, some of them may require modification; many of them, however, may be accepted as already definitely established not only for the systems upon which they are based, but for binary systems in general.

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CHAPTER IX

SOME BINARY SYSTEMS OF SPECIAL INTEREST

Having studied the orbit elements of the binary stars in their more general relations, it will be of interest next to consider the various systems in themselves, the extent, and the limitations, of our knowledge of their motions and physical conditions. Selection is here an obvious necessity, and in making my choice I have been influenced in part by the historical associations connected with certain systems, in part by the peculiarities of the orbit. Some of the systems are among those for which our knowledge is relatively full and exact; others present anomalies still more or less baffling to the investigator.

α CENTAURI

Our nearest known stellar neighbor, α *Centauri*, is a system of more than ordinary interest. One of the first half dozen double stars to be discovered—the very first among the stars of the southern heavens—it also divides with 61 *Cygni* the honor of being the first whose approximate distance, or parallax, became known. It consists of two very bright stars, 0.3 and 1.7 magnitude, respectively, which revolve in a strongly elliptic orbit so highly inclined to the plane of projection that at times they are separated by fully 22″, at others by less than 2″.

Accurate micrometer measures of relative position begin only with Sir John Herschel, in 1834, but meridian circle observations date back to Lacaille's time, 1752. Since these early dates the system has been observed regularly with meridian circle, micrometer, and heliometer, and the position of its components has been measured on photographic plates. The material is therefore ample for a very good determination of the orbit elements and of the proper motion of each component and excellent use has been made of it. The successive sets of elements by Roberts, See, Doberck and Lohse are in substantial accord and the orbit is probably as well known as that of any visual binary star. The parallax is known with equal precision; the value resulting from the excellent heliometer measures by Gill and Elkin would alone assure that; but this value has been confirmed by the accordant results by Roberts from the discussion of meridian circle observations and those by Wright from measures of the relative radial velocities of the components, to which reference has been made on an earlier page. The spectrograph has also given us the radial velocity of the center of mass of the system, −22.2 kilometers per second (in 1904).

Taking the orbit data in Table I of the preceding chapter, and the parallax, 0.76″, we find that the major semi-axis of the system equals 23.2 astronomical units; but since the eccentricity is 0.51, at periastron the components are separated by only 11.4, at apastron by fully thirty-five astronomical units; at the former time, that is, they are nearly as close together as *Saturn* and the Sun, and at the latter, farther apart than *Neptune* and the Sun.

Wright's measures of radial velocity were made in 1904 when the two components were near the nodal points. They showed that at that time the fainter component was approaching, the brighter one receding from the Sun, relatively to the motion of the center of mass of the system. The former was therefore at the descending, the latter at the ascending node in the relative orbit; and hence, on the system of notation adopted in this book, the angle 25°, given for Ω in the table of orbits, is the ascending node and the algebraic sign of the inclination is positive. *α Centauri* is therefore one of the very few visual binary systems for which the position of the orbit plane has been uniquely determined.

According to the adopted values for the parallax and orbit elements, the total mass of *α Centauri* is almost precisely twice that of the Sun, and all investigators of the proper motions of the two components agree that the brighter is very slightly the more massive of the two. Since the spectrum of this component is also practically identical with the solar spectrum, it is frequently referred to as a replica of the Sun. Further, if we assume 0.0 as the Sun's absolute magnitude (that is, its apparent magnitude were it removed to a distance corresponding to a parallax of 1″), at the distance of *α Centauri* it would shine as a star of +0.6 magnitude or somewhat less brightly than α_1 *Centauri*. Hence, if the luminosity per unit of surface area is the same in the two cases, as the similarity of the spectra would lead us to expect, α_1 *Centauri* must be rather larger and less dense than the Sun. The spectrum of α_2 *Centauri* is of later type, and this fact as well as its magnitude, 1.7, indicates that its luminosity is less than that of its primary; probably it is at least as dense as the Sun.

Finally, the accurate value of the proper motion of the system, 3.688″ (corresponding to a velocity of about twenty-three kilometers per second or a little less than five astronomical units per year), combined with the value of the radial velocity given above shows that the system is rushing through space with a velocity of thirty-two kilometers per second, which carries it about seven times the distance from the Earth to the Sun in a year. This is fully sixty per cent. greater than the motion of translation of our solar system.

SIRIUS

Several references have been made to *Sirius* on the earlier pages of this volume but it will not be amiss to give a more connected account of the star here. It was in 1834 that Bessel noticed that the proper motion of *Sirius*, the brightest star in the sky, was variable. Six years later he noted a similar phenomenon in the proper motion of *Procyon*, and by 1844 he had worked out the nature of the variation sufficiently to become convinced that it was due in each instance to the attraction of an invisible companion. His famous letter to Humboldt on the subject has often been quoted: "I adhere", he wrote, "to the conviction that *Procyon* and *Sirius* are genuine binary systems, each consisting of a visible and an invisible star. We have no reason to suppose that luminosity is a necessary property of cosmical bodies. The visibility of countless stars is no argument against the invisibility of countless others."

Peters examined the existing meridian circle observations in 1851 and concluded that they supported Bessel's hypothesis; ten years later, T. H. Safford repeated the investigation and "assigned to the companion a position angle of 83.8° for the epoch 1862.1." The most complete discussion, however, was that of Auwers, who "placed the question beyond doubt by determining the orbits and relative masses of the bright star and the invisible companion; but before the results were published, Mr. Alvan G. Clark discovered the companion, in 1862, near its predicted place." Bond's measures for the epoch 1862.19, in fact, placed the companion 10.07″ from the primary in the position angle 84.6°.

Since that time it has described more than an entire revolution and the orbit elements, now known with high precision, agree as well as could reasonably be expected with Auwers's, computed before the companion's discovery. The revolution period, for instance, is 49.32 years according to Lohse; Auwers's value was 49.42 years. The eccentricity of the true orbit is greater than that for the orbit of *α Centauri*, but the inclination of the orbit plane is considerably less and the apparent ellipse is therefore a more open one, the maximum apparent separation of the components being about 11.2″ and the minimum a little less than 2″. The bright star is so exceedingly brilliant, however, that it is impossible to see the faint companion with any telescope when it is near its minimum distance. Thus, periastron passage occurred early in 1894, but the last preceding measure was Burnham's in the spring of 1892 when the angular separation was 4.19″, and the little star was not again seen until October, 1896, when my first measure gave an apparent distance of 3.81″.

The magnitude of *Sirius*, on the Harvard scale, is −1.58; estimates of the brightness of the companion vary, but it is probably not far from 8.5 on the same scale, a difference of 10.1 magnitudes. Accepting these figures, *Sirius radiates more than 11,000 times as much light as its companion;* and, if the parallax 0.376″ is correct, fully thirty times as much as our

Sun. Yet, according to the best mass determinations, the bright star is only 2.56, the companion 0.74 times as massive as the Sun; and Adams finds that the small star, which to the eye seems decidedly the yellower, has the same spectrum (Class A) as the bright star. These are facts which, as Campbell says, "we are powerless to explain at present."

KRUEGER 60

The system known as Krueger 60 (the closer pair was really discovered by Burnham in his careful examination of all the double stars noted by Krueger in the course of his meridian circle observations) offers a strong contrast to the two we have been considering, not only in its appearance but in many of its physical characteristics, though like them it is remarkable for its large proper motion and its large parallax. *Sirius* and *α Centauri* are two of the brightest stars in the sky and are also of great absolute brilliance; Krueger 60 is only of the ninth magnitude, despite its large parallax, and is among the feeblest of known stars in its actual radiating power. The orbit elements of the former two are known with accuracy; the companion star in Krueger 60 has been observed through less than one-third of a revolution, and there is, moreover, a most unfortunate gap in the series of measures from 1890 to 1898. It is therefore surprising that we have any orbit for it at all. The star was neglected at first because the average pair of ninth magnitude stars with a separation of 2.32″, such as Burnham's measure in 1890 gave for Krueger 60, does not change perceptibly in a century, and the exceptional character of this pair was not recognized until Doolittle measured it again in 1898. Since that time Barnard and Doolittle have measured it systematically. An excellent idea of the telescopic appearance of the system and of the rapidity of the orbital motion is given by the photographs taken in 1908 and 1915 by Professor Barnard, who has kindly permitted me to reprint them here. The photographs also show a third star which is independent of the binary and is being left behind by the latter in its motion through space.

Plate V. Photographs of Krueger 60, in 1908 and in 1915

Though the orbit elements are necessarily rather uncertain, Russell has been able greatly to limit the range of possible solutions by a skilful use of the dynamical relations connecting the observed coordinates (position angle and distance) and the times of observation. Thanks to this and to the series of measures connecting the star A of the binary with the independent star, and to our very precise knowledge of the value of the parallax, which three unusually accordant determinations by Barnard, Schlesinger and Russell fix at $+0.256''$, our acquaintance with the physical conditions in the system is far more complete than so short an observed arc would ordinarily make possible. The period given in my table of orbits may indeed be in error by as much as eight years, the eccentricity by one-third or more of its whole amount; nevertheless, the mass of the system is very well determined, the value, 0.45 ⊙ being correct probably to within ten per cent. The mass ratio is somewhat more uncertain and may lie anywhere between 0.36 and 0.56; that is, the mass of the brighter star is from 3/10 to 1/3, the mass of the fainter, from 1/6 to 1/8 as great as the Sun's. The latter is the smallest mass so far established with any degree of probability for any star.

The estimates of the magnitudes of the components vary considerably, but we may adopt 9.3 and 10.8 as approximately correct (Russell adopts 9.6 and 11.3). Now the Sun at the distance of Krueger 60 would shine as a star of the third magnitude, hence the two stars have actual luminosities only 1/330 and 1/1320 that of the Sun. According to Adams, the spectrum of the pair is of the Class Mb. This is undoubtedly the spectrum of the brighter star and the fainter one probably has a spectrum even more advanced. There is thus little question but that the mean density of each star is much greater than the Sun's and the intensity of its radiation per unit of surface area much smaller, and we may agree with Russell that these two stars are nearing "the very end of their evolutionary history."

I may point out that our knowledge of this system is due primarily to the fact that it is, relatively speaking, so very near our own. The parallax of $+0.256''$ corresponds to a distance of about 12.7 light years. But the average star of the apparent ninth magnitude is more nearly 400 light years distant, say thirty times as far away. Remove Krueger 60 to such distance and its components become nearly 7½ magnitudes fainter; that is 16.8 and 18.3 magnitude, respectively. The system would then be invisible as a double star in any existing telescope, and the probability of its detection on photographs taken with our giant reflectors would be extremely small.

ζ CAPRICORNI AND 85 PEGASI

Table I of Chapter VIII contains the orbits of ten systems which have revolution periods ranging from fifteen to twenty-seven years, as well as four of still shorter period, δ *Equulei*, Ho 212, κ *Pegasi*, and A 88. These systems have nearly all been discovered in recent years; only one, 42 *Comae Berenices*, dates back to Struve's time and one, δ *Equulei*, to the time of Otto Struve; seven of them were discovered by Burnham.

Several of these systems have already been referred to in more or less detail and it will suffice to describe briefly two others which are fairly typical of the group and which at the same time present some interesting contrasts. These are ζ *Capricorni* and 85 *Pegasi*. The former has a period of 21.17 years, a small eccentricity, 0.185, and fairly high orbit inclination, 69.4°. Since the major semi-axis is 0.565″, the two components are well separated when at, or near, the extremities of the rather narrow apparent ellipse and are then easily measured; but at minimum separation, when the angular distance is only 0.2″, measures are very difficult, particularly from stations in the northern hemisphere. As in all the binaries of this group, there is no apparent deviation from simple elliptic motion and the orbit elements are well determined. My orbit, computed in 1900, still represents the observed motion with precision although the stars have traversed an arc of nearly 270° since then. The proper motion is small for so bright a star, less than 0.03″ annually, and the parallax has not been determined. If we assume the mass to be twice that of the Sun, the parallax will be +0.06″; if the mass is eight times the solar mass, the parallax is +0.04″. Probably these figures may be regarded as approximate limits and we may therefore assume that each of the two components is from ten to twenty-five times as luminous as our Sun and that the orbit, in its dimensions, is comparable to *Saturn's*. The spectral class is A2, hence the stars probably exceed the Sun in surface brightness, but are probably also larger and less dense than the Sun.

The orbit of 85 *Pegasi* is not quite so determinate as that of ζ *Sagittarii* because the great difference in the magnitudes of the components makes measures difficult even when the apparent separation has its maximum value of about 0.8″, and practically impossible near the time of periastron passage when the angular distance is only 0.25″. Nevertheless, the orbit computed by Bowyer and Furner in 1906 represents the motion as observed to date within the limit of accidental error of measure, and we may regard the period at least as well determined. Though the eccentricity, 0.46, and the orbit inclination, 53.1°, are of average value, the fact that the line of nodes is nearly perpendicular to the major axis of the true orbit makes the apparent ellipse rather an open one.

The system has the large proper motion of 1.3″ annually, and according to Kapteyn and Weersma the parallax is +0.067″. Combining the latter value with the apparent magnitude, 5.8 on the Harvard scale, we find that the bright star is almost precisely equal to the Sun in absolute magnitude; and since it has a spectrum of Class G0, we should expect the surface conditions in the two bodies to be similar. The spectrum of the companion is not known, but this star certainly radiates more feebly than its primary for it gives out less than 1/100 as much light (apparent magnitude 11.0). Now the interesting fact is that the independent investigations of Comstock, Bowyer and Furner, and Boss agree in making the smaller star from two to four times the more massive, though Boss adopts equal masses because he regards the meridian circle measures as of small weight and the result they give as *a priori* improbable. But we have in *Sirius* a system in which an even greater disparity between mass and luminosity in the two components is beyond question and it is by no means impossible that a similar relation holds in the system of 85 *Pegasi*. The orbit elements and parallax give a total mass 2.65 times that of the Sun, hence, if the brighter star is really the less massive of the two, its effective radiating power must exceed that of the Sun though the two bodies give out light of the same spectral characteristics.

ε HYDRAE AND ζ CANCRI

Reference has already been made to the multiple systems ε *Hydrae* and ζ *Cancri* and to the fact that there is a considerable number of triple, quadruple, and multiple systems among the stars known primarily as double stars. Such systems raise many interesting questions; as for example, whether it is possible to detect the influence of the more distant star or stars upon the orbital motion of the closer pair, or, conversely, the effect of the binary pair upon the observed motion of the other stars.

As a matter of fact, it is a disturbing force of the latter kind that has actually been noticed in one of the two systems named, ε *Hydrae*, while in ζ *Cancri* (and other pairs), an explanation for observed irregularities in the motion of the distant star has been found in the existence of an invisible fourth star. In each of these two systems the larger, or at least the brighter, star is the one which has been divided into a close binary, designated for convenience as AB. The third star, C, in ζ *Cancri* has shown an annual relative motion of about 0.5° in an arc which is concave toward AB. The motion of C in ε *Hydrae* is of similar character, but slower; and in each case there are periodic irregularities in the motion such that when the observed positions are plotted they lie on a curve showing more or less definite loops at intervals of about eighteen years in the one case, and of fifteen years in the other. Seeliger's analysis leaves no question but that in the system of ε *Hydrae* the irregularity is only apparent, being caused by the fact that the effective light center of the system AB describes a small ellipse by virtue of the orbital motion in this fifteen-year period binary. In the system of ζ *Cancri*, on the other hand, it seems to be real and to be due to the presence of a fourth star, invisible in the telescope, which revolves with C about a common center in a slightly eccentric orbit with a period of

17.6 years. In neither system, and, in fact, in no other visual triple, has it been possible to detect any disturbing effect, due to the more distant third star, in the orbital motion of the closer binary pair; but it must be remembered that the unavoidable errors of observation are large in comparison with the possible perturbations.

μ HERCULIS AND 40 ERIDANI

Frequently it is the smaller, or fainter star of a wide pair which is itself a close binary. Two of the most interesting systems of this kind are *μ Herculis* and 40 *Eridani*. These are both bright stars, of magnitude 3.48 and 4.48, respectively, on the Harvard scale, with companions, noted as 9.5 and 9.2 by Struve, separated from their primaries by 32″ and 82″, respectively. Struve describes each of these wide pairs as "yellow and blue," and the color of the bright stars harmonizes with the spectral Class G5, assigned to them in the *Revised Harvard Photometry*. But, according to Adams, the 'blue' companion to *μ Herculis* belongs to Class Mb, whereas the equally 'blue' companion to 40 *Eridani* belongs to Class A2! Evidently, the color contrast observed in such pairs is not a safe guide to difference of spectral class; and if it is not such in wide pairs like these, how much less is it to be trusted in closer pairs!

The companion of *μ Herculis* was first noted as double by Alvan Clark in 1856, having escaped the search both of Herschel and of Struve. Since discovery, it has described more than a complete revolution, and the period and other orbit elements are quite definitely established, presenting no unusual features. The bright star, A, has an annual proper motion of 0.817″ in 203.35° and the binary pair, BC, is certainly moving through space with it, for the measures of AB since Struve's time show very little relative motion. Assuming that the parallax determined for A also applies to BC, I have given the mass of the binary, in Table IX of the preceding chapter, as just equal to that of the Sun. On the same assumption, the semi-major axis of the orbit has a length of twelve and one-quarter astronomical units, which is greater than that of

Saturn's orbit, while the eccentricity is about the same as that of the orbit of *Mercury*.

The double companion to 40 *Eridani* forms a system drawn on a larger scale; its period is 180 years, so that since its discovery by the elder Herschel it has not had time to complete a full revolution. The elements, however, are fairly determinate and present the remarkable feature of the smallest eccentricity established in any visual binary. One would not expect to find this associated with a period of 180 years, but the pair is an easy one to measure with even moderately good telescopes and the measures since about 1850 are plentiful. The fact therefore seems to be beyond doubt. The bright star, A, has the exceptionally large proper motion of 4.11″ annually in 213.3° and the faint pair is travelling with it, for as in the system of μ *Herculis*, the measures of AB indicate little relative motion. The comparative nearness to the Sun which would be inferred from this large proper motion has been confirmed by direct measures of the parallax. It is hardly possible to question the resulting mass and absolute magnitudes of the two components and yet it is most remarkable to find stars of such feeble luminosity belonging to the spectral Class A2.

On the basis of the assumed orbit and parallax, the semimajor axis has a length of 27.5 astronomical units and the orbit is therefore nearly as large as Neptune's, while the binary is 470 astronomical units from the bright star. This star is about seven-tenths of a magnitude fainter than the Sun would be if viewed from the same distance, but its spectral class is a little later, indicating somewhat feebler luminosity. We may therefore assume that its mass is equal to the Sun's and that the period of revolution of the binary about the bright star will be about 7,000 years, if the present separation, 83″, is the mean distance. However faulty these figures may be, there is no doubt at all but that in these triple systems we have reproductions on a vast scale of the Earth-Moon-Sun type of orbital motion, making due allowance for differences of relative mass in the components; indeed, we may expect to find systems of dimensions even greater than these.

This raises a question. The mere fact that two stars have the same motion through space is ordinarily held to be sufficient evidence of the binary character of any double star; but to what degree of apparent separation does this criterion hold good? Small stars are known which have the same proper motion as brighter ones 30′ or more away,[1] which, in the case of even moderate remoteness as stellar distances go, may correspond to an actual separation of half a light year or more. Certainly the two stars are physically connected and probably they have had a common origin; but does that imply orbital motion in the ordinary sense, or shall we simply say that they move through space along parallel paths as the stars in the *Taurus* cluster or those in the *Ursae Major* cluster do? This is one of the questions to which no general answer can be given at the present time.

POLARIS

A triple system quite different in type from those we have been discussing is the one of which the North Star, *Polaris*, is the only visible component. Six plates taken by Campbell with the Mills spectrograph in 1896, gave radial velocities for *Polaris* ranging only from −18.9 to −20.3 kilometers per second. As the plates were taken at varying intervals between September 8 and December 8, they seemed to furnish sufficient evidence of a constant velocity; but when additional plates were taken in August 1899, the first three gave velocities of −13.1, −11.4, and −9.0 kilometers, respectively. "Inasmuch," writes Campbell, "as a range of four kilometers is not permissible in the case of such an excellent spectrum, the star was suspected to be a short period variable," and plates were promptly secured on a number of additional nights. These settled the question, showing that the radial velocity has a

[1] The faint star (11.0 visual, 13.5 photographic magnitude) 2° 13′ from *α Centauri*, for which Innes proposes the name *Proxima Centauri*, has practically the same proper motion and *parallax* as the bright star. The great angular separation, by reason of the exceptionally large parallax, corresponds to a linear separation which is only about twenty-two times that between 40 *Eridani* and its binary companion; but, even so, if the star is moving in an orbit with *α Centauri*, the period of revolution must be measured in hundreds of thousands of years! The large color index denotes extreme redness, and, intrinsically, it is by far the faintest star of which we have a definite knowledge.

range of six kilometers, the period of one complete oscillation being slightly less than four days. It appeared, upon investigation, that the six plates of 1896 were taken "at intervals differing but little from multiples of the period of the binary system and therefore fell near the same point in the velocity curve."

The period 3.968+ days represents the observations in 1896 and also the very numerous ones made from 1899 to the present time, and there is no evidence of any variation in this element nor in the value of K in the binary system. But the velocity of the center of mass for this system was about -17.2 kilometers in 1896.75, whereas it was only -11.5 in 1899.75. Such a discrepancy could not possibly be attributed to errors of observation or measurement and were rightly regarded by Campbell as clear evidence that *Polaris* is at least a triple system, the four-day period binary moving in a much larger orbit with a third star, invisible to us. At one time it was thought that this larger orbit had a period of about twelve years, but this is not the case. The maximum velocity (minimum negative velocity) in this orbit seems to have been reached in 1899 or 1900; in 1910.5 it was about -15.8, in 1916.2, -17.8 kilometers. What the minimum value (maximum negative velocity) will be cannot now be predicted, but it is apparent that the orbit is quite eccentric with a period exceeding twenty years. The ratio of more than 1,800 to 1 shown by the long period oscillation to that of short period is far greater than the ratio of the periods in any known visual triple.

Campbell has described the spectrum as 'excellent'; examination of several of the plates shows that the absorption lines are numerous and well defined. They are not broad nor yet hazy, but compare very favorably with the lines in the solar spectrum as seen in the light reflected by *Venus*, and the probable error of measure from a single plate is therefore less than half a kilometer. Though the range in velocity is small, the character of the spectrum places it beyond doubt; and that this variation is due to orbital motion in a binary system was until quite recently questioned by no one, for at the time of its discovery and for many years thereafter *Polaris* was regarded as "perhaps the star in all the sky of whose constancy in light we may be most certain." Indeed, it had been adopted by Pickering as the standard star (magnitude 2.12 in the *Revised Harvard Photometry*) in the extensive photometric work carried out at Harvard College Observatory, and had held the same position in similar researches elsewhere.

In 1911, however, Hertzsprung was able to show that it was really a variable star, the light curve resembling, in general, that of the Cepheid variables. The period was found to be identical with that of the velocity variation, the range in light (photographic), 0.17 magnitudes. Examination of the extensive photometric data at Harvard confirmed the discovery, as did the observations made elsewhere, and it was shown, as Hertzsprung had anticipated, that the range of light, visually, is only about one-tenth of a magnitude, thus adding another point of resemblance to the Cepheid variables which, as a class, show a greater range of light variation in the light of short wave-length than in the light which most strongly affects the eye.

Now the spectrum, and the characteristics of the four-day orbit of *Polaris* had already been recognized as strikingly similar to those of the known Cepheids; in fact, this was one of the reasons for Hertzsprung's investigation. We must certainly, therefore, class *Polaris* among the Cepheids and must face the question whether we shall give up the well-founded belief that it is a short period binary system simply because we now discover that its light varies in a particular manner in the same period as its velocity. Frankly conceding that no theory so far advanced for the cause of the light variation, on the assumption that it is somehow due to the interaction of the two components in a binary system, is wholly acceptable, I am still of opinion that we have no reason to abandon our faith that it *is* a binary until some substitute theory is brought forward *which will account for the periodic displacement of the spectral lines*. All theories so far advanced, on the hypothesis that in a Cepheid variable we are dealing with physical changes in the atmosphere of a single star, fail to explain this line displacement. Moreover, they call for broadened, and probably hazy absorption lines instead of the well-defined and quite narrow lines which characterize the spectra of *Polaris* and many other Cepheids.

CASTOR

The spectroscopic binaries α_1 and α_2 *Geminorum*, which form, respectively, the fainter and the brighter component of the well-known double star, *Castor*, present an interesting contrast in the forms of their orbits. Curtis's definitive investigation shows that the orbit of α_1 (discovered by Belopolsky in 1896, period 2.928285 days) is practically circular, the rigorous least squares solution giving the eccentricity 0.01; but the orbit of α_2 (discovered by Curtis in 1904, period 9.218826 days) has an eccentricity 0.5033, above the average value for the *visual* binaries. The one, then, is typical, the other exceptional.

Unfortunately, the elements of the visual binary cannot yet be regarded as determinate though *Castor* was, as we have seen, the first stellar system for which orbital motion was definitely established. The latest and most thorough research relating to this orbit is the one by the veteran computer, W. A. Doberck, the man who has investigated more double star orbits than any other astronomer. He gives three alternative sets of elements with periods respectively, 268,347, and 501 years, but regards the 347-year period as the most probable. Recent observations seem to support this conclusion, and Curtis has adopted it in his speculations concerning the system. The relative radial velocity of the two visual components, derived from the latter's investigation of the spectroscopic binary orbits was 7.14 kilometers. This, with Doberck's elements, gives a parallax of $+0.05''$ and a total mass 12.7 times that of the Sun. Moreover, if the two spectroscopic binaries are revolving in the same plane as the visual system, for which Doberck finds the inclination, 63.6°, the semi-major axes of the two systems are:

α_1 *Geminorum*, a = 1,435,000 kilometers
α_2 *Geminorum*, a = 1,667,000 kilometers

that is, they are of the same order of magnitude. Therefore, to account for the relative periods, it is necessary to assign to the fainter star a mass about six times as great as that of the brighter one. Finally, on the generally accepted theories of stellar evolution, the difference of eccentricity means that the brighter and less massive system is the older; the fainter system, with circular orbit, comparatively, of recent origin. These anomalous results are, of course, at present almost entirely speculative. Curtis finds no evidence of irregularities in the velocity curves, nor of light variation in either component of the visual pair.

δ ORIONIS

An inspection of the elements of δ *Orionis* would not lead to the impression that it was distinguished in any particular manner from the other short period spectroscopic binaries listed in Table II. It has, however, several points of interest. The faintest of the three stars in the Belt of *Orion*, it is one of the spectroscopic binaries investigated by Stebbins with the selenium photometer, the measures definitely establishing the fact that it is a variable star with a light range of 0.15 magnitudes of which 0.08 magnitudes is due to eclipses. The two stars of the system are probably ellipsoidal in form and keep always the same face turned toward each other. In other words, under the action of powerful tidal forces the rotation period of each has been brought to, or kept in equality with the period of its orbital revolution. The light curve also indicates that the surface brightness of the disks is not uniform, each body being "brighter on the front side in its motion in the orbit." The mean mass of the system is determined as 0.006 that of the Sun, the larger star having a radius at least five, and the smaller one a radius at least 1.4 times that of the Sun.

It is, however, a discovery made by Hartmann from the observations of the star's spectrum at Potsdam in the years 1900 to 1903 that gives the system its peculiar position among spectroscopic binaries. The spectrum is of Class B and the calcium lines (known as the H and K lines) are narrow and sharply defined, while the other lines, chiefly due to helium and hydrogen, are more or less diffuse. Now Hartmann's measures of the hydrogen and helium lines indicated a range in radial velocity of about 200 kilometers per second, but those of the calcium lines gave a nearly constant velocity! From a series of plates taken at Allegheny in the years 1908 to 1912, Jordan has arrived at similar results, the H and K lines giving a nearly constant velocity whose mean is +18.7 kilometers whereas the range from the other lines is almost precisely 200 kilometers.

The velocity of the center of mass of the system, as derived by Jordan from lines not due to calcium is +15.2 kilometers, differing from the mean for the calcium lines by 3.5 kilometers, and Hartmann's measures indicated an even greater discrepancy, 7 kilometers. This may be evidence that the material producing the calcium absorption does not belong to the star.[2] Take into account two other facts, (1) that the constellation of *Orion* is well known as a region of space containing widespread nebulosity, and (2) that the Sun's own motion through space is carrying it away from that region with a velocity of about 18 kilometers per second, and it is apparent that Hartmann's assumption of the existence between us and δ *Orionis* of a cloud of calcium vapor stationary in space (so far as radial velocity is concerned) has much to commend it. It is entirely possible that the explanation might have won general acceptance if δ *Orionis* had remained the only star showing this anomaly. But the number has gradually increased until now some twenty or twenty-five binary systems are known in which the H and K lines yield either constant velocities or velocities which have a different range—generally in the sense of being much smaller—from that derived from the other lines of the spectrum.

Every one of these stars is a 'helium star', that is one belonging to Spectral Classes Oe to B2; and many astronomers have asked the question pertinently raised by Young, "Why should the calcium clouds always lie in front of a star of type B2 or earlier?" In his paper on the orbit of χ *Aurigae*, a star for which the H and K lines give a velocity range approximately

[2] It is not impossible that the discrepancy is due, in part at least, to errors in the wavelengths assumed for some of the standard lines.

half as great as that for the other lines, Young summed up the other known facts relating to systems of this type. Omitting details and specific illustrations, these are essentially as follows:

In eight stars the calcium lines give a velocity that remains nearly constant and that differs somewhat, as a rule, from the velocity of the system determined from the other lines which in every instance show a large range in velocity.

"In several stars the calcium lines are known to vary differently from the other lines." Orbits have been computed from the H and K lines for four such stars, in two instances with amplitudes about half those given by the remaining lines. Young believes that future investigation will reveal all gradations in the variation of the calcium lines from constant velocity to oscillations equal to those of the other lines in the spectra; and, indeed, several stars are already known in which there is not much difference in the mean velocities from the calcium and from the other lines.

"There seems to be no exception to the rule that when the calcium lines are sharp and narrow and the other lines broad, the star exhibits a variable radial velocity." This relation was first announced, I believe, by Frost, who has used it in predicting, with success, that certain stars with spectra of the character described would prove to be spectroscopic binaries.

It is very difficult to harmonize all of these facts with the theory of stationary calcium clouds independent of the systems, and therefore at least two other theories have been advanced. One of these, that the phenomena are due to anomalous dispersion effects may be passed with the mere statement that it has not won wide acceptance. The other, that the calcium vapor in question envelops one or both stars, lying high above the effective photosphere, is far more plausible. So far as I am aware, this theory was first put in definite form by Lee in his discussion of the orbit of 9 *Camelopardalis*. He shows, as does Young in the paper already cited, that it will not only account for the phenomena observed in the particular system under investigation, but that it is sufficiently elastic to meet the varying demands made upon it by conditions in other systems. However, the theory at best is as yet only a working hypothesis, which must meet the tests of many further applications before it can be regarded as definitely established.

β AURIGAE

The binaries which imprint the spectra of both components upon the plates are of special interest, because the difference in the range of oscillation shown by the two sets of lines permits us to determine the mass ratio, while their intensities afford us a measure of the relative brightness of the two components. That there is a close correspondence between difference of mass and difference of brightness is a relation to which Schlesinger, Baker and others have called attention.

In connection with Table VIII of the preceding chapter, in which are listed all systems for which I could find published values of the mass ratio, I commented upon the fact that nearly all of these stars belong to Spectral Classes B and A. In his paper on α *Virginis*, in 1909, Baker made the comment that, as a rule, they also have short periods and nearly circular orbits. If the stars in Table VIII are examined with respect to these two elements, it will be found that two-thirds of them have periods under ten days, and that two-thirds (not always the same stars) have eccentricities under 0.08.

One of the most interesting stars of this class is β *Aurigae*, the second spectroscopic binary star in point of discovery. The two components are of nearly equal brightness and have identical spectra, of Class A. Baker defines the spectra more precisely by saying that "they are further advanced than that of *Sirius* and are intermediate to those of the components of *Castor*." The magnitude of the star on the Harvard scale is 2.07, hence each component is about 2.8 magnitude; which means, if we adopt Kapteyn and Weersma's parallax, 0.014″, that each is about 6½ magnitudes brighter than our Sun would be if viewed from the same distance. Their radiating power per unit of surface area doubtless greatly exceeds that of the Sun, but even so they must be vastly larger than the Sun and probably much less dense.

The system has been the subject of extensive investigations by the astronomers at the Harvard, Potsdam and Pulkowa Observatories, but the most recent and most complete discussion is Baker's, based primarily upon the spectrograms secured at the Allegheny Observatory in 1908–1909 but utilizing also the results of the earlier researches. From Baker's definitive elements it appears that the two masses are $2.21/\sin^3 i$ and $2.17/\sin^3 i$, the Sun's mass being the unit, and that the mean distances of the two components from the center of mass of the system are respectively $5,934,000/\sin i$, and $6,047,000/\sin i$ kilometers. The value of i, the inclination, is of course unknown, but in view of the range of 220 kilometers in the radial velocity it is almost certainly as great as 30°; and since sin 30° = 0.5 we may say that the superior limit to the two masses is probably about seventeen times the Sun's mass, and to the linear distance between the centers of the components, not more than 24,000,000 kilometers. The actual values may be decidedly smaller. Now 24,000,000 kilometers is less than one-sixth the distance from the Earth to the Sun, hence it is evident, when we recall the parallax given above, that no telescope can show the system as a double star; even were the inclination as low as 6°, the angular separation would be barely 0.01″.

Baker's analysis of the Allegheny observations gave strong evidence of systematic departures from elliptic motion which could be represented as a secondary oscillation with a period one-third that of the primary. This is a phenomenon frequently noted in spectroscopic binary star orbits and has received special attention at the Allegheny Observatory. Schlesinger has shown that it is sometimes merely a "blend effect" due to the unrecognized presence of a faint spectrum produced by the companion star. This broadens the lines with the result that the measures give velocities displaced at any given point toward smaller values. In *β Aurigae* no such effect can operate for the orbit "is derived exclusively from measures of the separated component lines." A comparative study of the Potsdam, Pulkowa and Allegheny measures of velocity shows, however, that the secondary oscillation has no physical basis, that is, that it is not due to any perturbation in the stars' motions, for the range of the oscillation varies with the instrument employed, being "greatest for the spectrograph giving the least dispersion and the smallest separation of the two spectra." There are doubtless cases in which such oscillations have a physical basis, as I have remarked on an earlier page, but the fact that in β *Aurigae*, in 30 H *Ursae Majoris* and in other systems subjected to a searching analysis the oscillations have been found to lie in the measures rather than in the star's motion warns us against accepting the reality of such appearances until the most thorough tests have been applied.

One of the most striking results developed by Baker's discussion is that the period of revolution is apparently slowly lengthening. Intercomparison of the observations at Potsdam in 1888–1897 and 1903–1904, at Pulkowa in 1902–1903 and at Allegheny in 1908–1909, shows a progression from 3.95993 days for the epoch 1896.4 to 3.960029 days for the epoch 1906.1, an increment of +0.000010 days, or +0.86 seconds annually. Similar variations in period, *always in the sense of a slow increase with the time*, have been found in three other spectroscopic binaries which are not eclipsing variables; in ε *Herculis* by Harper, in λ *Andromedae* by Burns, and in θ *Aquilae* by Baker. Granting the uncertainties that may attach to the numerical values and even to the fact of the increase in one or two of the systems, the results as they stand lead Baker to the query, "Are we not here actually observing the progress of evolution from the spectroscopic to the visual binary?" This is a point of the greatest interest, but we must regard the query as speculative only until further observations have established the fact of progressive increase of period beyond question. If a series of spectrograms of these stars taken at the present time, and another series taken, say, five or ten years later, show that the progression continues, we may accept it as strong evidence of a definite advance in the evolution of the stars.

ALGOL

β *Persei*, better known as *Algol*, has been the subject of many memoirs which are counted among the classics of astronomical literature, but our knowledge of the mechanism of the system is still far from satisfactory. The general character of the light variation, it is true, was established by Goodericke, its discoverer, as early as 1783, and his hypothesis as to its cause has been fully confirmed by the spectrograph; but even now we cannot regard the light curve as definitively established, and as measures of the radial velocity accumulate, we find ever further evidence of complexities in the system. An adequate account of the work that has been done on this star and of the theories that have been offered in explanation of the observed phenomena would require a chapter. The present note must be limited to a description of some of the more striking facts which have been developed in these researches.

Argelander first demonstrated the existence of fluctuations in the period between successive light minima, and Chandler's more extensive studies, utilizing all available observations from Goodericke's time to 1888, not only confirmed this conclusion but led him to explain them as arising from a long period inequality which he ascribed to the presence of a third body in the system revolving with the eclipsing binary in a practically circular orbit in a period of about 130 years. Chandler's theory predicted a slow increase in the period of the eclipsing binary (*i. e.*, in the length of the interval between successive light minima) beginning with the closing years of the Nineteenth Century and continuing until late in the Twentieth, and this is apparently supported by Stebbins whose period for the variable, derived from observations made in 1909–1910, is six seconds longer than Chandler's for the years 1871 to 1888. Chandler, moreover, supported his hypothesis by an analysis of the proper motion of *Algol*, finding evidence of variations similar in character (but of much longer period) to those which led to Bessel's predictions with respect to *Sirius* and *Procyon*.

While the depth and duration of the primary minimum (loss of light equals about 1.2 magnitudes, length of eclipse, from first contact to fourth, about ten hours of the 68.8 hour period) had been defined with satisfactory precision at least as long ago as Schönfeld's time, no decisive evidence of variation in the normal brightness between these minima was forthcoming until 1909–1910. Stebbins's measures with the selenium cell photometer then established the existence of a secondary minimum of only 0.06 magnitude, and also indicated that the light is not strictly constant at any phase, the maximum brilliancy falling not half way between the two minima, as we might expect, but just before and just after the secondary minimum. While Stebbins's measures are of remarkable accuracy, he would be the first to say that it is highly desirable to have his results, especially for the minute variation between minima, confirmed by additional measures with the new photo-electric cell photometers.

Pickering, in 1880, had worked out the dimensions of the system in terms of the (unknown) linear separation of the two components, on the eclipse hypothesis; Vogel, in 1889, not only put this hypothesis beyond question by his demonstration that the star is a spectroscopic binary, but also from his measures of the radial velocities determined the diameters of the two stars, as well as the dimensions of the orbit, in terms of kilometers, making the assumption, however, that the two components are of equal mass. At the time, Vogel's work seemed to leave little to be desired, but in more recent years Schlesinger and Curtiss have found that the light minima lag from one and one-half to two hours behind the time demanded by the spectrographic measures on the eclipse theory. The ephemeris based upon Stebbins's more accurate light curve, it is true, has removed 1 h. 16 m. of this discrepancy; but since the Allegheny observers find a similar lag in the case of several other Algol variables it must still be regarded as a matter demanding investigation.

In 1906, Belopolsky announced a long-period oscillation in the radial velocities of *Algol*, and this was confirmed by Schlesinger in 1912. The latter finds the period to be 1.874 years, the semi-amplitude (K) 9.14 kilometers, and the orbit nearly circular, and his examination of the photometric material from 1852 to 1887 reveals evidence of a corresponding oscillation in the times of light minimum.

Inadequate as this summary is, it is yet sufficient to show how great is the complexity of the system, and to indicate the necessity for further accurate photometric and spectrographic measures for the formulation of a complete theory. So far as the eclipsing binary is concerned, we can, however, give a description of the principal features that is fairly reliable. The following account is based upon Stebbins's results, but whereas he gives figures for several alternative assumptions, I shall give only those which seem to me to be nearest to the truth.

The companion to *Algol* probably rotates upon its axis once in every revolution and thus, like our Moon, keeps the same face turned toward its primary, and this face is brighter than the opposite side by reason of the radiation received from *Algol*. The primary may also rotate once in a revolution but of this we have no evidence. Possibly the two bodies are slightly ellipsoidal in form and, as Shapley suggests, may possess more or less extensive absorbing atmospheres which produce a gradual darkening of the disks toward the edge, or limb. Assuming the stars to be spherical and without such absorbing atmospheres, the light curve gives 82.3° for the inclination of the orbit plane. It is a reasonable assumption, also, in view of their relative brightness, that *Algol* is twice as massive as its companion. If we adopt this ratio and the value 1,600,000 kilometers for $a \sin i$, we have the following figures for the dimensions, masses and densities, the Sun being taken as the unit:

Radius of *Algol* = 1.45
Radius of companion = 1.66
Mass of *Algol* = 0.37
Mass of companion = 0.18
Density of *Algol* = 0.12
Density of companion = 0.04

The distance between the centers of the two stars is 4.77 times the radius of *Algol* and the mean density of the system 0.07 that of the Sun.

The stellar magnitude of *Algol* is 2.2; the light of the faint hemisphere of the companion equals that of a star of 5.2 magnitude, the light of the brighter hemisphere that of a star of 4.6 magnitude. If the value +0.029″, adopted for the parallax by Kapteyn and Weersma, is correct, the light of *Algol* is

160 times that of the Sun, the light of the faint and bright hemispheres of the companion, respectively ten and seventeen times that of the Sun. If the value $+0.07''$ adopted in some earlier discussions of the system is correct, the luminosities are reduced to about one-sixth of those just given. We may safely regard the latter as the minimum limits and may therefore say that even the faint side of the 'dark' companion to *Algol* is far more brilliant than our Sun.

It was inevitable that in this chapter more emphasis should be placed upon the systems in whose study we encounter difficulties than upon those which apparently conform to the laws of motion in a simple Keplerian ellipse. The chapter might be extended indefinitely; systems like 70 *Ophiuchi*, ϕ *Persei*, β *Lyrae*, to name no more, present problems and puzzles which are still unsolved. It is precisely the unexplained anomalies, the irregularities in the motions, the apparent contradictions among the spectral lines, the complexities of various kinds, that offer the best opportunities for discoveries which may advance our knowledge of the forces at work in the stellar systems. If it were not for these, our interest in the systems themselves would soon flag; but as the case stands there is always some new problem to spur us on, "for each stellar system," as Miss Clerke says, "is in effect a world by itself, original in its design, varied in its relationships, teeming with details of high significance."

An excellent illustration is found in the system of κ *Pegasi*, now under investigation at the Lick Observatory by Dr. F. Henroteau. Dr. Campbell, in 1900, discovered that one of the components of the well known 11.3-year period visual binary was itself a binary with a period of about six days. It now develops that the spectrum of the *other component in the visual system* is also present upon the plates, thus making it possible to determine the parallax, the mass of the system, the true inclination of the orbit plane, the linear dimensions of the triple system, the relative masses of the visual components, and an independent orbit of the 11.3-year period binary. Comparison of the plates taken in the present year with the earlier ones already shows the variation of the velocity of the center of mass of the six-day period binary, due to its motion in the larger orbit, and gives strong indications of a revolution of the line of apsides, a perturbative effect suspected in several other short period spectroscopic binaries, but not, so far as I am aware, established beyond question in any case.[3]

REFERENCES

It is unnecessary to give a list of the many papers which have been consulted in the preparation of this chapter. The references in earlier chapters will suffice for the visual systems and Campbell's 'Third Catalogue of Spectroscopic Binary Stars', now in preparation, will give the references for all spectroscopic binaries.

[3] Dr. Henroteau's investigation of the system of κ *Pegasi* has been published as *Lick Observatory Bulletin*, Number 304.

CHAPTER X

A STATISTICAL STUDY OF THE VISUAL DOUBLE STARS IN THE NORTHERN SKY

The Lick Observatory double star survey referred to in my historical sketch was undertaken with the definite purpose of accumulating data for the statistical study of the number and distribution of the visual double stars. I sought answers to such questions as these: What is the number of double stars relatively to the number of all stars to a given magnitude? Is the ratio the same for faint stars as for bright ones, in one part of the sky as in another, for stars of one spectral class as for stars of other classes? The answers which the survey affords for the stars in the northern half of the sky will be considered in the present chapter.

The data consist of all visual double stars as bright as 9.0 B. D. magnitude which fall within the distance limits set by the following 'working definition' of a double star proposed by me in 1911:

(1) Two stars shall be considered to constitute a double star when the apparent distance between them falls within the following limits:

1″ if the combined magnitude of the components is fainter than 11.0
3″ if the combined magnitude of the components is fainter than 9.0 B. D.
5″ if the combined magnitude of the components lies between 6.0 and 9.0 B. D.
10″ if the combined magnitude of the components lies between 4.0 and 6.0 B. D.
20″ if the combined magnitude of the components lies between 2.0 and 4.0 B. D.
40″ if the combined magnitude of the components is brighter than 2.0 B. D.

(2) Pairs which exceed these limits shall be entitled to the name double star only when it has been shown (*a*) that orbital motion exists;

(*b*) that the two components have a well defined common proper motion, or proper motions of the 61 *Cygni* type; (*c*) that the parallax is decidedly greater than the average for stars of corresponding magnitude.[1]

In all, there are 5,400 pairs, distributed according to the discoverer as follows:

W. Struve	1053
O. Struve	296
Burnham	551
Hough	237
Hussey	766
Aitken	2057
Miscellaneous	440

The chief contributors to the 'miscellaneous' list are Jonckheere, Espin and observers of the Astronomische Gesellschaft Star Catalogue. Sir John Herschel's long lists contribute only twenty-eight pairs and Sir William Herschel's discoveries are included in the number credited to W. Struve. It should be noted that a given system is counted only once though it may have three or even four or more components. In such triple systems as ζ *Cancri*, γ *Andromedae*, and ϵ *Hydrae*, the closer pair is listed; in an occasional system in which the close pair is very faint the wider pair is the one counted. A number of the stars credited to Burnham and later observers also have Σ or $o\Sigma$ numbers which are disregarded in the above tabulation.

The first question to consider is whether the data are homogeneous, for it is obvious that they can make no claim to be exhaustive. However carefully an observer may work, some pairs which he might discover with a given telescope will surely escape him. His eye may be fatigued, unnoticed haze or momentary bad seeing may blur out a faint companion star,

[1] The definition, with correspondence relating to it, will be found in the *Astronomische Nachrichten* (*188*, 281, 1911). Comstock and E. C. Pickering there suggest limits based upon the apparent magnitude, the former using the formula $s = c\left(\frac{2}{7}\right)^m$, the latter, the formula, $\log. s = c - 0.2\,m$, where s is the distance in seconds of arc between the components, m, the apparent magnitude, and c an arbitrary constant. If the values of c in the two formulae are so chosen as to give the limit 5.0″ for stars of magnitude 6.0, the formulae will give the limits 0.75″ and 1.25″, respectively, for stars of 9.0 magnitude. From the theoretical point of view either formula gives more logical limits than the ones in my definition, but there were practical considerations, fully stated in the article referred to, which led to the adoption of the latter.

or it may chance that at the date of examination the two components are so nearly in conjunction as to be below the resolving power of the telescope. The number of known double stars can only be regarded as the lower limit to the number which might be discovered. Homogeneity was earnestly sought for, care being taken to work only when in good physical condition and when the seeing was good, the practical test being the power to recognize very close and difficult pairs at a glance. But variations in the conditions are inevitable when the working program requires years for its execution and doubtless such variations have affected the present results. Careful comparison, however, shows no discernible difference in the thoroughness of the work done at different seasons of the year or in different parts of the sky, and it may fairly be said that the results of the survey represent the capacity of the combination of telescope and observer under average good atmospheric conditions at Mount Hamilton. If the work had all been done with the thirty-six-inch refractor the resulting data might be considered quite homogeneous. Unfortunately, a considerable part of it, including practically the entire area north of $+60°$ declination, was done with the twelve-inch telescope, and it becomes necessary to consider the relative efficiency of the two instruments.

I have applied two tests: first, the comparison of the most difficult pairs discovered with each instrument; second, the re-examination with the thirty-six-inch of some 1,200 stars previously examined with the twelve-inch telescope. I find that, under the usual observing conditions, a pair with nearly equal components separated by only $0.15''$, or a companion star as faint as 14.5 magnitude and not less than $1.5''$ from its primary is practically certain of detection with the thirty-six-inch; with the twelve-inch, the corresponding limits in the two cases are $0.25''$ and 13 to 13.5 magnitude. Twelve new double stars were added by the re-examination of the 1,200 stars. From these tests, taking into account the proportion of the whole work done with the twelve-inch telescope, I conclude that about 250 pairs would have been added if the entire northern sky had been surveyed with the thirty-six-inch.

According to Seeliger's count of the B. D. stars there are 100,979 as bright as 9.0 magnitude in the northern hemisphere. Of these, 5,400, or 1 in 18.7 on the average, have actually been found to be double within the limits set above. If we add only 200 pairs, the ratio becomes 1 : 18.03. A definite answer is thus given to my first question: "*At least one in every eighteen, on the average, of the stars in the northern half of the sky which are as bright as 9.0 B. D. magnitude is a close double star visible with the thirty-six-inch refractor.*" There is no reason to doubt that the ratio is equally high in the southern half of the sky.

TABLE I

The Distribution of Double Stars in Right Ascension and Declination

R.A.	0^h to 1^h	2^h to 3^h	4^h to 5^h	6^h to 7^h	8^h to 9^h	10^h to 11^h	12^h to 13^h	14^h to 15^h	16^h to 17^h	18^h to 19^h	20^h to 21^h	22^h to 23^h
Decl.												
0°–9°	6.3	6.3	7.4	6.0	6.0	5.9	5.5	6.4	5.4	6.0	5.3	4.9
10–19	5.4	5.4	6.4	6.2	5.5	5.5	4.8	6.1	4.5	6.0	5.4	5.2
20–29	5.2	5.6	5.9	5.1	4.8	6.4	5.2	4.8	5.4	6.2	4.9	4.2
30–39	6.0	5.5	5.9	4.4	4.3	4.8	5.4	4.1	5.0	5.6	5.2	4.8
40–49	6.7	5.0	5.0	4.5	4.5	6.2	3.2	5.6	5.5	4.2	5.2	4.9
50–59	6.2	6.2	8.0	5.2	4.8	3.6	4.4	6.0	4.6	4.4	6.8	4.9
60–69	4.6	5.0	7.1	2.9	4.3	4.4	2.6	4.1	3.6	4.1	5.3	6.8
70–79	5.9	3.8	4.2	3.5	3.1	2.6	2.8	2.2	3.4	4.9	5.0	6.5
80–89	(0^h to 5^h)		3.7	(6^h to 11^h)		3.3	(12^h to 17^h)		3.4	(18^h to 23^h)		3.9

The figures give the percentages of double stars among stars to 9.0 B. D. magnitude; the average percentage for the whole northern sky is 5.35.

Table I exhibits the distribution of the 5,400 double stars in right ascension and declination as compared with the distribution of the B. D. stars to 9.0 magnitude, the figures giving the percentage of double stars in each area. There are obvious irregularities in the table but no evidence of systematic differences that can be regarded as seasonal effects. The percentages are as high in the sky areas surveyed in winter as in those surveyed in summer. There is a falling off in the percentage in the high declinations, especially in the regions well removed from the Milky Way, which is doubtless due in part to the fact that the area north of 60° was almost entirely surveyed with the twelve-inch telescope. The broken line in the table represents very roughly the position of the central line of the Milky Way, and it will be noted that the percentages near this line are, in general, above the average.

The distribution with respect to the plane of the Milky Way is more clearly brought out when the stars are tabulated according to galactic latitude. This has been done in Tables II and III, in the former of which the stars are divided in classes according to magnitude and the latitudes into zones each 20°

TABLE II

The Distribution of Double Stars by Magnitude Classes and Zones of Galactic Latitude

Mag. to	m 6.5	m 6.6–7.0	m 7.1–7.5	m 7.6–8.0	m 8.1–8.5	m 8.6–9.0	Total
Zone							
I	19	13	14	29	40	84	199
II	43	28	50	68	114	193	496
III	60	43	56	79	148	254	640
IV	96	54	81	132	232	401	996
V	121	88	133	249	376	653	1,620
VI	84	51	81	134	221	395	966
VII	28	23	18	54	90	154	367
VIII	7	6	5	12	31	55	116
Total	458	306	438	757	1,252	2,189	5,400

wide, beginning at the north galactic pole. Zone V therefore includes the area from +10° to −10° galactic latitude, and Zone IX, which ends at the south galactic pole and lies entirely below the equator, is not represented. As was to be expected, the numbers in every column of this table are largest in Zone V and fall to minima in Zones I and VIII. The question is whether this condensation toward the Milky Way is greater than that of all the stars. Table III provides the answer.

TABLE III

Density of Double Stars by Magnitude Classes and Galactic Latitude Compared with the Density of B.D. Stars to 9.0 Magnitude (after Seeliger).

Mag.	to 6.5		6.6–7.0		7.1–7.5		7.6–8.0		8.1–8.6		8.6–9.0	
Zone	B.D.	D.S.	B.D.	D.S.	B.D.	D.S.	B.D.	D.S.	B.D.	D.S.	B.D.	D.S.
I	0.551	0.395	0.431	0.374	0.518	0.266	0.404	0.304	0.419	0.261	0.382	0.325
II	.572	.456	.445	.410	.497	.484	.424	.351	.441	.390	.404	.380
III	.639	.480	.554	.474	.599	.407	.509	.307	.512	.380	.484	.377
IV	.790	.789	.689	.610	.765	.606	.730	.529	.720	.614	.728	.613
V	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
VI	.912	.822	.787	.687	.842	.721	.772	.639	.799	.694	.789	.716
VII	.572	.395	.427	.446	.467	.231	.480	.370	.521	.407	.527	.401
VIII	.428	.307	.315	.361	.352	.199	.373	.255	.462	.435	.527	.445

Since the zones are not of equal area, and since only the first lies wholly in the northern hemisphere, the fairest comparison is that afforded by the relative densities per square degree of double stars and all stars of the corresponding magnitudes. The double star densities were determined by dividing the figures in Table II by the number of square degrees in each zone area; the figures were then reduced to a common standard by making the density in each column unity in Zone V, the Milky Way Zone. Seeliger has published corresponding data for all of the B. D. stars and the two sets of values are entered in Table III in the columns D. S. and B. D., respectively. It is clear that the density curves of double stars rise to sharper maxima in the Zone V than the corresponding curves of stars in general do.

This fact is exhibited in a more striking manner if we tabulate, as in Table IV, the percentages of double stars in five areas, the Milky Way Zone, the 20° zone on either side of it and the areas north of +30° and south of −30° galactic latitude.

TABLE IV

Percentages of Double Stars

Galactic Latitude	B.D. Stars to 9.0	Double Stars	Percentage of Double Stars
+90° to +30°	26,948	1,335	4.95
+30 +10	19,355	996	5.15
+10 −10	26,477	1,620	6.13
−10 −30	17,831	966	5.13
−30 −70	10,368	483	4.66

The increased percentage in Zone V must be accepted as real. Table III shows that stars of all magnitude classes participate in it, and an examination of my charts leads to the conclusion that it cannot be an observing effect, for some areas of all galactic latitudes were examined in summer, others in winter; the area north of +60° declination, examined almost exclusively with the twelve-inch, extends from −3° to +27°

TABLE V

The Distribution of Double Stars in Galactic Latitude by Distance Classes

Dist.	0.00″ to 0.50″		0.51″ to 1.00″		1.01″ to 1.00″		1.51″ to 2.00″		2.01″ to 3.00″		3.01″ to 4.00″		4.01″ to 5.00″		5.01″ and over	
Zone	No.	%	No.	%	No.	%	No.	%	No.	%	No.	%	No.	%	No.	%
I	41	11	31	10	25	14	24	16	39	17	15	9	14	10	10	20
II	101	26	92	30	56	30	56	38	61	26	65	40	48	33	17	34
III	139	36	99	32	90	49	65	44	101	43	61	37	53	36	32	64
IV	225	58	175	57	131	71	104	70	133	57	112	68	74	51	42	84
V	388	100	306	100	184	100	149	100	233	100	164	100	146	100	50	100
VI	247	64	152	50	119	65	105	70	144	62	101	62	72	49	26	52
VII	82	21	77	25	52	28	36	24	42	18	32	20	29	20	17	34
VIII	33	8	18	6	14	8	12	8	12	5	9	6	12	8	6	12

galactic latitude and the areas of high galactic latitude, both north and south, were examined mainly with the thirty-six-inch refractor. We may therefore say that *close visual double stars are relatively more numerous in the Milky Way than elsewhere in the sky.*

Since the stellar system quite certainly extends to a much greater distance in the plane of the Milky Way than at right angles to it, the natural inquiry is whether the increased double star density in the Milky Way is not merely a perspective effect. If this were the case it would seem that we might expect a relatively higher percentage of very close pairs in Zone V than of pairs of moderate separation. Table V, however, in which the 5,400 pairs are grouped according to galactic latitude and angular separation, shows that the percentage increase toward Zone V is as great for stars with angular distances of from 2″ to 5″ as for stars under 1″.

Let us consider next the relation between the angular separation and magnitude. This is shown in Table VI where the stars are arranged with these qualities as arguments. The sums of the numbers in the first two columns of the table are entered in the third, thence the numbers are given for uniform steps in angular distance to the final column. Every line of columns three to seven exhibits a marked increase in the number of pairs as the angular distance diminishes.

This is still more clearly brought out when the figures in these five columns are expressed as percentages of the total number of stars under 5″ separation in each magnitude class. Now it is generally believed that stars of a given magnitude, for example from 8.6 to 9.0, are approximately at the same distance from us, on the average. Hence the observed increase in the number of pairs as the angular distance diminishes is not a mere perspective effect, but represents *a real increase in the number of pairs as the orbital dimensions diminish.* The uniformity of the percentages in each column in the lower division of the table and the fair agreement between the ratios of the figures in each line of the first two columns in the upper division both argue that the Lick Observatory survey was as thorough for the fainter stars as for the brighter ones. This was to be expected, for with the thirty-six-inch refractor a close and difficult double of magnitude eight to nine is as readily seen as a similar pair of brighter magnitude; in fact, I find that 123 of the 379 pairs with angular separation of

TABLE VI

The Distribution of Double Stars by Angular Distance and Magnitude

Dist.	to 0.50″	0.51″ to 1.00″	0.00″ to 1.00″	1.01″ to 2.00″	2.01″ to 3.00″	3.01″ to 4.00″	4.01″ to 5.00″	5.01″ and over
Mag.								
≧ 6.5	75	63	138	83	62	41	31	99
6.6–7.0	82	52	134	59	42	40	21	14
7.1–7.5	103	67	170	99	64	48	31	29
7.6–8.0	178	132	310	164	107	85	63	26
8.1–8.5	310	223	533	285	173	128	111	21
8.6–9.0	508	413	921	532	317	217	191	11
Totals	1,256	954	2,206	1,222	765	559	448	200

Percentages

≧ 6.5			39	23	17	12	9	100
6.6–7.0			45	20	14	14	7	100
7.1–7.5			41	24	16	12	7	100
7.6–8.0			42	22	15	12	9	100
8.1–8.5			43	23	14	11	9	100
8.6–9.0			42	25	14	10	9	100
Totals			42	23	15	11	9	100

0.25″ or less, and 385 of the 877 pairs with angular separation between 0.26″ and 0.50″ discovered in that survey are of B. D. magnitude between 8.6 and 9.0.

These statements are of significance also in connection with the figures exhibited in Table VII which shows the percentage of double stars of each magnitude class. The high percentage in the first line is partly due to the relatively large number of bright double stars wider than 5.0″. If this number is reduced to the same order as that of the wider pairs for the other magnitude classes, the percentage becomes 9.1.

TABLE VII

Percentage of Double Stars by Magnitude Classes

Magnitude	B.D. Stars	Double Stars	Percentage of Double Stars
to 6.5	4,120	458	11.1
6.6–7.0	3,887	306	7.9
7.1–7.5	6,054	438	7.2
7.6–8.0	11,168	758	6.8
8.1–8.5	22,898	1,251	5.5
8.6–9.0	52,852	2,189	4.1

In view of what has been said above it is impossible to attribute the disparity in the percentage of double stars among the brighter and fainter stars to incompleteness of the data, and this becomes even more apparent when we recall that the distance limit adopted (5.0″) is the same for the stars in the last magnitude class as for the stars as bright as 6.0. If the adopted limit were a function of the magnitude, like those suggested by Comstock and Pickering,[2] and the value 5.0″ were retained for stars of 6.0 magnitude, we would have practically the same percentages in the first two lines of Table VII, and diminished values in the following ones, the final percentage (for stars 8.6 to 9.0) being only half its present value.

The very high percentage of spectroscopic binary systems among stars as bright as 5.5 magnitude may or may not be significant in this connection, for we do not yet know whether that percentage will hold among the fainter stars, but on the evidence before us we may venture the suggestion that perhaps the stars of larger mass, and hence presumably greater luminosity, are the ones which have developed into binary systems.

[2] See footnote, page 253.

The stars have been grouped in the foregoing tables without regard to their spectral class. We have now to see whether any of the results obtained vary with the star's spectrum. In June of the present year (1917) Director E. C. Pickering and Miss Annie J. Cannon, of the Harvard College Observatory, generously permitted me to compare my list of double stars with the manuscript of the *New Draper Catalogue of Stellar Spectra*, which contains the spectral classification of over 200,000 stars. It is due to their great kindness, for which I cannot adequately express my thanks, that I am able to present the results which follow. The comparison with the *New Draper Catalogue* has provided the spectral classification for 3,919 of the 5,400 visual double stars in the northern sky. Of the remaining 1,481 stars only 15 are as bright as 8.0 magnitude (and about half of those were inadvertently omitted from the list I took to Cambridge), 218 lie between 8.1 and 8.5, and 1,248 between 8.6 and 9.0 magnitudes. Practically, then, the data as to spectral class are complete for stars to 8.0, nearly complete for stars to 8.5, but less than half complete for the stars between 8.6 and 9.0 B. D. magnitude.

The *New Draper Catalogue* is based upon objective prism spectra and therefore, in general, does not record separately the spectra of the components of such double stars as are considered here. The spectrum for these is either the spectrum of the bright component, or, if the magnitudes are nearly equal, a blend of the spectra of the two components. A very large number of the closer double stars have components which differ but little in brightness, and it has long been known that such pairs exhibit little or no color contrast. Indeed, Struve, in his discussion of the stars in the *Mensurae Micrometricae* showed that the color contrast between the components of a double star was a function of their difference of magnitude, and Lewis and other writers have extended his researches. In my own observing I have noted the components of hundreds of pairs of nearly equal magnitude as being of the same color. I may add that although my eyes are entirely normal as regards color perception I have never been able to see such violent contrasts in any pair of 'colored stars' as are scopic binaries evidently show a marked preference for the early spectral classes from Bo to Ao, the visual binaries for the classes from Ao to Ko. The maxima in certain of the sub-classes such as Ao and Ko are apparently real but probably without special significance. The most unexpected feature of the table is that the number of spectroscopic binaries of the M sub-classes is relatively much greater than that of the visual binaries.

To study the relations of these groupings to those of the stars in general, I have combined the sub-classes into six larger classes, using the Harvard system, which is indicated by the heavy vertical rules in Table VIII, and have compared the results with Pickering's tabulation of the stars in the *Revised Harvard Photometry*.

TABLE IX

The Spectral Classes of the Binary Stars and of the Stars in the Revised Harvard Photometry

Numbers

	B	A	F	G	K	M	All
Vis. Bin. (all)	157	1,251	532	1,093	837	49	3,919
Vis. Bin. (to 6.5)	59	142	65	83	93	18	460
Spec. Bin.	198	161	61	71	95	19	605
R.H.P. Stars.	822	2,755	1,097	932	2,531	636	8,773

Percentages

	B	A	F	G	K	M	All
Vis. Bin. (all)	4	32	14	28	21	1	100
Vis. Bin. (to 6.5)	13	31	14	18	20	4	100
Spec. Bin.	33	26	10	12	16	3	100
R.H.P. Stars.	9	21	13	11	29	7	100
V. B. (all) −H.P.	−5	+1	+1	+17	−8	−6	
V. B. (6.5) −H.P.	+4	0	+1	+ 7	−9	−3	
Sp. B. −H.P.	+24	−5	−3	+ 1	−13	−4	

The numbers are not strictly homogeneous even for the *Revised Harvard Photometry* stars and the spectroscopic binaries, but are sufficiently so to permit comparison. To make the data for the visual binaries more directly comparable with the rest, I have given separately the numbers for the systems as bright as 6.5 magnitude B. D. The upper part of Table IX records the actual numbers of the stars of each category and of each spectral class; the lower part, the percentages and the excess or defect of each percentage when compared with that of the *Revised Harvard Photometry* stars. The figures in the last two lines may be affected to a certain degree by the lack of strict homogeneity in the data, but they certainly place beyond question the fact that spectroscopic binaries as bright as 5.5 magnitude are far more numerous among Class B stars than among stars of other spectral classes, and that the visual binaries as bright as 6.5 magnitude are in excess among Class G stars and least numerous among stars of Classes K and M.

The discovery of spectroscopic binary stars is quite independent of the distance of the system, for the displacement of the lines in the spectrum (disregarding variations in the physical condition of the star) depends simply upon the velocity of the light source in the line of sight. The observed or angular separation of a visual binary, on the other hand, is a function of its distance from us, and it is necessary to consider whether this fact may not seriously influence the distribution shown in the table, for many different recent investigations have led astronomers to believe that stars of Classes F and G are, on the average, the nearest, those of Classes B and M the most remote. It is not easy to estimate the effect of this factor. Certainly the 5″ limit imposed in collecting the data does not favor the Class G stars at the expense of those of Classes B and M, for it is clear that of two systems of the same linear dimensions the more distant one might fall within the limit whereas the nearer one might be excluded by it. The more distant pair might, it is true, fall below the resolving power of the telescope while the nearer pair would be well enough separated to be discovered, but this would not account for the fact that the percentage of visual binaries (to 6.5 magnitude) is greater among the Class B stars than among those of Class F or Class K. Admitting the uncertainties arising from the unknown distances of the double stars and from errors in the classification, I am still of the opinion that the last two lines of the table give a fair *qualitative* representation of the true distribution with respect to spectrum of the visual binaries among stars as bright as 6.5 magnitude, as well as of the spectroscopic binaries.

The percentages for the visual binaries to 9.0 magnitude will quite certainly be modified when we can compare the

TABLE X

The Distribution of Double Stars by Spectral Class and Galactic Latitude

	B	A	F	G	K	M	Un-known	B.D. Stars
I	1	11	24	51	52	2	58	4,276
II	1	45	50	159	121	3	117	8,798
III	4	87	78	170	129	5	167	13,874
IV	16	260	95	186	148	8	283	19,355
V	109	542	118	202	168	14	467	26,477
VI	22	247	101	183	122	11	280	17,831
VII	3	48	48	105	74	5	84	7,998
VIII	1	11	18	37	23	1	25	2,370
Sums	157	1,251	532	1,093	837	49	1,481	100,979

Percentages

I	1	2	20	25	31	14	12	16
II	1	8	42	79	72	21	25	33
III	4	16	66	84	77	36	36	52
IV	15	48	81	92	88	57	61	73
V	100	100	100	100	100	100	100	100
VI	20	46	86	91	73	79	60	67
VII	3	9	41	52	44	36	18	30
VIII	1	2	15	18	14	7	5	9

3,919 double star spectra with the spectra of all the stars in the *New Draper Catalogue*. It may be pointed out also that the 9.0 magnitude limit may operate here to favor Classes B and A at the expense of Class K, for the former classes contain a high percentage of stars intrinsically bright, the latter a considerable percentage of 'dwarfs'.

Table X is arranged to show the distribution of the visual binaries of different spectral classes with respect to the galactic plane. I have placed the systems for which the spectral class is unknown in the seventh column of the table and in the final column have added Seeliger's count of the B. D. stars to 9.0 magnitude.

Visual binaries of every spectral class increase in number as we approach the Milky Way, but only in Classes B, A and M (and in the systems of unknown spectrum) is the increase

TABLE XI

The Distribution of the Spectroscopic Binaries by Spectral Class and Galactic Latitude

	B	A	F	G	K	M	All
I	0	7	1	4	0	0	12
II	3	21	5	4	3	1	37
III	6	17	6	11	7	1	48
IV	31	27	9	4	19	3	93
V	76	44	12	25	20	9	186
VI	63	26	17	16	20	3	145
VII	10	11	7	6	17	1	52
VIII	7	8	3	1	8	0	27
IX	2	0	1	0	1	1	5
Sums	198	161	61	71	95	19	605

greater than that shown by the B. D. stars to 9.0 as a whole. The fact brought out in Tables III and IV that visual double stars are relatively more numerous in the Milky Way than elsewhere in the sky must therefore be due to the systems of these classes, and especially, in view of the actual numbers, to those of Class A and of unknown spectral class. Possibly the latter are also largely of Class A. The strong concentration of the Class M stars toward the Galaxy is due entirely to the thirteen stars of Sub-classes Ma and Mb. These fall in Zones IV, V and VI in the numbers 3, 7 and 3, respectively. The thirty-six stars of Sub-class K5 are as uniformly distributed as those of Class K (G5 to K2).

Table XI, exhibiting the distribution in galactic latitude of the 605 spectroscopic binaries, shows a similar increase in the

TABLE XII

The Distribution of Double Stars by Magnitude and Spectral Class

	B	A	F	G	K	M	B.D. Stars
To 6.0 mag.	38	85	38	53	61	39	4,120
6.1–6.5	21	57	27	30	32	6	
6.6–7.0	15	109	51	59	72	1	3,887
7.1–7.5	20	162	64	104	82	3	6,054
7.6–8.0	30	232	126	209	137	8	11,168
8.1–8.5	21	308	128	347	220	9	22,898
Sums	145	953	434	802	604	66	48,127

Numbers Expressing Relative Frequency

To 6.5 mag.	87.4	14.1	15.3	7.4	12.9	149.6	(11.1)
6.6–7.0	23.6	11.4	12.8	5.5	10.6	4.7	(7.9)
7.1–7.5	20.4	11.9	10.3	6.3	7.8	6.9	(7.2)
7.6–8.0	16.5	8.5	11.0	6.8	7.0	9.6	(6.8)
8.1–8.5	5.5	5.5	5.5	5.5	5.5	5.5	(5.5)

numbers in all spectral classes as we approach the Milky Way. In this case we are considering the entire sky and may therefore bring out the effect of the Milky Way more strikingly by dividing the sky into three zones by lines at $\pm 30°$ galactic latitude. The central zone contains half the sky area and it is seen that six-sevenths of the Class B systems, four-fifths of the Class M, but only five-eighths of those of each of the remaining classes fall within this half. Class B stars as a whole are relatively more numerous in the Milky Way than elsewhere in the sky in a ratio as great as that shown by the spectroscopic binaries of this class, but stars of Class M are quite uniformly distributed over the sky. That so high a percentage both of the visual and of the spectroscopic binaries of this class are in or near the Milky Way is somewhat surprising.

In Table VII it was shown that the percentage of double stars is higher among the bright than among the faint stars. To see whether this holds for all spectral classes, I have arranged Table XII.

The spectral class is known for less than half of the double stars of magnitude 8.6 to 9.0 and this group is therefore omitted from the table. The upper part of the table gives the actual numbers of pairs of the various classes; the lower part was formed as follows: The numbers in each row in the upper part (after combining the first two lines) were divided by the number of B. D. stars of corresponding magnitude. The figures in the six resulting columns were then made comparable by introducing factors to make those in the bottom row all equal to 5.5, the percentage of double stars among stars of magnitude 8.1–8.6 in Table VII.

The comparison of the first six columns in this part of the table with the figures in parenthesis in the last column, which are repeated from Table VII, shows that double stars of every spectral class except G contribute to the increase in the percentage of pairs as the apparent brightness increases. The excess of double stars brighter than 6.5 magnitude in Classes B and M is very striking, but the actual numbers as given in the upper part of the table show that in the other magnitude grades double stars of Classes A, F and K are more effective in producing the observed progression.

I have prepared tables similar to Table VI for the double stars of each spectral class, but for the sake of brevity I shall give only their summary.

In forming the percentages I have, as in Table VI, omitted the pairs wider than 5.0″. It is apparent that the progression of numbers with decreasing angular distance begins to fall off with Class K, and is still less appreciable in Class M. The individual tables of which XIII is the summary indicate that the distribution by magnitude shown in Table VI holds in Classes A, F, and G, and also in Classes B and K, though in these it is less marked for the brighter stars. It does not hold for the small group of Class M stars.

TABLE XIII

The Distribution of 3,919 Visual Binaries by Spectral Class and Angular Separation

	0.00″ to 1.00″	1.01″ to 2.00″	2.01″ to 3.00″	3.01″ to 4.00″	4.01″ to 5.00″	over 5.00″
Class B	60	33	29	10	9	16
A	540	283	158	122	88	60
F	222	115	55	74	44	22
G	489	236	143	106	82	37
K	273	187	144	108	78	47
M	12	10	8	5	6	8

Percentages

B	43	23	21	7	6	100
A	45	24	13	10	8	100
F	43	23	11	14	9	100
G	46	22	14	10	8	100
K	34	24	18	14	10	100
M	29	25	29	12	15	100

A large percentage of the visual binary stars have measurable proper motions and are therefore among the stars relatively near us (say within a radius of 500 light-years) those in which rapid orbital motion has been observed being, on the average, the nearer ones. No investigation of their actual distribution in space, however, has so far been published; nor has the spatial distribution of the spectroscopic binaries as a whole been investigated.

Hertzsprung, however, has studied the distribution of the Cepheid binaries, and Russell and Shapley that of the eclipsing binaries. In each case it is found that the majority of the stars investigated (sixty-eight Cepheids and ninety eclipsing systems) lie within a region bounded by planes drawn parallel to, and comparatively close to the plane of the Milky Way, and that within this region they are scattered over a vast extent of space. In the case of the Cepheids the median plane of the region passes about 123 light-years south of the Sun, the mean distance of a system from this plane is 296 light-years, while their distances from a central point defined by a perpendicular from the Sun upon the median plane range from about 500 to nearly 10,000 light-years. The median plane for the eclipsing binaries lies about 100 light-years south of the Sun, a considerable majority of the systems, including nearly all those of short period, lie within 500 light-years from this plane and all of the others within 1,000 light-years; whereas their distances parallel to the plane from a central point defined as before range up to at least 8,000 light-years provided that light suffers no absorption in its passage through space. Assuming what the authors consider "a plausible amount of absorption of light in space," the limits are cut down to 4,000 light-years.

It is almost unnecessary to add that both of these conclusions, however reasonable they may be regarded, rest upon more or less speculative hypotheses and are not to be considered as definitely established.

CHAPTER XI

THE ORIGIN OF THE BINARY STARS

Any theory of the origin of the binary stars must take account of the facts of observation which have been outlined in the three preceding chapters. Chief among these are the following:

1. The great number of the binary systems. On the average, at least one star in eighteen of those as bright as 9.0 magnitude is a binary visible in our telescopes; at least one in every three or four of those as bright as 5.5 magnitude is a binary revealed by the spectrograph. These are *minimum* values. Our knowledge of the fainter stars is very incomplete, but on the evidence before us we may safely say that one-third, probably two-fifths, of the stars are binary systems; some astronomers, indeed, are inclined to think that systems of the type of our solar system may be the exception rather than the rule.

2. The considerable percentage of systems with three or more components. It is well within the truth to say that one in twenty of the known visual binaries has at least one additional member either visible in the telescope or made known by the spectrograph, and many systems are quadruple or still more complex. Evidence is also accumulating to the effect that many of the purely spectrographic systems are triple or multiple. It is a fact of undoubted significance that, as a rule, triple systems, whether visual or spectroscopic, consist of a close binary pair and a companion relatively distant.

3. The close correlation between the length of period, or size of system, and the degree of ellipticity in the orbit. The visual binaries, with orbit periods to be reckoned in tens of years, have average eccentricity close to 0.5; the spectroscopic binaries, with periods to be counted in days or fractions of a day, have average eccentricity of about 0.2; and in each class the average eccentricity increases with the average length of the period. We have, then, an unbroken progression or series of orbits from systems in which the two components revolve in a fraction of a day in circular orbits and practically in surface contact, to systems in which the components, separated by hundreds of times the distance from the Earth to the Sun, revolve in

highly elliptic orbits in periods of hundreds, perhaps thousands of years.

4. The correlation between length of period and spectral type. The short period spectroscopic binaries are prevailingly of spectral Class B; the longer period spectroscopic binaries are usually of much later spectral class; the visual binaries have few representatives among stars of Class B and are most numerous among stars of Classes F and G.

5. The correlation between the relative brightness of the components and their relative mass. When the two components are of equal brightness they are of equal mass, so far as our investigations have been able to go; in other systems the brighter star is, almost without known exception, the more massive, but the range in mass is far smaller than the range in brightness. No system is known in which the two components have a mass-ratio as small as 1/10.

6. The relatively great mass of a binary system, taking the Sun as standard. Among the spectroscopic binaries showing two spectra we have found only one (θ *Aquilae*) which has a *minimum* mass value less than the Sun's; the other minimum values given in Table VIII of Chapter VIII, range from one and one-half to thirty times the solar mass. The average mass of the visual binaries for which the data are really determinate is nearly twice that of the Sun, and only one system is known (Krueger 60) in which the mass can with any probability be said to be as small as half that of the Sun.

7. The spectroscopic binary of Class B is, on the average, fully three times as massive as the binary of later spectral class. We have no data for the mass of a visual binary of Class B.

8. A few systems *are exceptions to the general rules*. For example, the short period spectroscopic binaries of early spectral class with very eccentric orbits; the long period spectroscopic binaries of late spectral class, and the long period visual binaries with nearly circular orbits; the occasional system in which the fainter star is apparently the more massive.

This enumeration is not exhaustive; I have, for instance, omitted the facts, brought out in my statistical study of the visual binaries in the northern sky, that such systems are somewhat more numerous, relatively, among the stars of the Milky Way than among those remote from it; that the binaries which cause this apparent concentration are chiefly of spectral classes B, A, and M; that there are apparently more binaries, relatively, among the brighter stars than among the fainter ones, to 9.0 magnitude; for some of these relations may prove to have no physical significance but to arise from the fact that our material necessarily consists of selected stars. As it stands, however, the enumeration is ample to indicate the difficulties attending any attempt to formulate a theory of general applicability.

If we consider a single binary system, it is conceivable that it might originate in any one of at least three different ways:

1. Two stars, hitherto independent, might approach each other under such conditions that each would be swerved from its original path and forced to revolve with the other in orbits about their common center of gravity (Capture Theory).

2. A single star, in its primal nebulous stage, or possibly even later, might divide into two, which would at first revolve in surface contact (Fission Theory).

3. The material in the primal nebula might condense about two nuclei separated by distances of the order of those now existing between the centers of the component stars (Independent Nuclei Theory).

No one of these theories is entirely satisfactory when we consider the binary stars as a whole, and this is sufficiently demonstrated by the fact that each one of the three has its advocates among able astronomers at the present time. Since I have no new theory to advance I shall content myself with an exposition of the principles involved in each of the three and its apparent accordance or discordance with the facts of observation.

What I have called the capture theory appears to have been proposed originally by Dr. G. Johnston Stoney, in 1867. On May 15 of that year he presented a paper to the Royal Society of London *On the Physical Constitution of the Sun and Stars*, the major portion of which is concerned with the Sun. In Section 2 of Part II, however, he treats of multiple stars and, distinctly stating that his deductions are "of necessity, a speculation," argues that if two stars should be brought very close to each other one of three things would happen. The third is that "they would brush against one another, but not to the extent of preventing the stars from getting clear again." In this event his analysis indicates that the stars would thereafter move in elliptic orbits, but that their atmospheres would become engaged at each periastron passage. Since the atmosphere of each star "is not a thing of uniform density," the resistance would take the form of forces acting, some tangentially, and some normally, to the stellar surfaces; the former would tend to reduce the periastron distance, the latter to increase it, and as one or the other dominated in a special case, the two stars would ultimately "fall into one another" or "gradually work themselves clear of one another." In the latter event a double star would result.

Meanwhile, "the heat into which much of the *vis viva* of the two components has been converted will dilate both to an immense size, and thus enable the two stars gradually in successive perihelion passages to climb, as it were, to the great distance asunder, which we find in the few cases in which the final perihelion distance can be rudely estimated, a length comparable with the intervals between the more remote planets and the Sun. During this process, the ellipticity of the orbit is at each revolution decreasing; but if the stars succeed in getting nearly clear of one another's atmospheres before the whole ellipticity is exhausted, the atmospheres will begin to shrink in the intervals between two perihelion passages more than they expand when the atmospheres get engaged, and will thus complete the separation of the two stars. When once this has taken place, a double star is permanently established."

According to Stoney, then, the eccentricity of a system decreases and its major axis increases up to the time when the atmospheres of the two stars are completely disengaged even at perihelion passage. The most favorable case for the formation of a double star on this hypothesis is presented when the original stars are of equal mass; and since no double star can result unless the unequal pressure of the atmospheres in their grazing collision has imparted to at least one of the two "a swift motion of rotation," Stoney imagines that under special conditions the rotational motion might become so great as to exceed the cohesive strength of the star and there might "result two or more fragments spinning violently," ultimately leading to the formation of triple systems like that of γ *Andromedae.*

I have given Stoney's theory somewhat in detail because the fundamental principle in it, namely, that collisions of stars in various degrees, central, partial, or grazing, might produce the stellar systems as we now know them has frequently appeared in later speculations on cosmogony. Professor A. W. Bickerton has elaborated it into a complete theory of cosmic evolution; it was regarded favorably by Lord Kelvin (so far as the binary stars are concerned); it appears again in Arrhenius's book, *Worlds in the Making*, and has quite recently been presented, with apparent approval, by Moulton in his discussion of the origin of binary stars in the revised edition of his text-book, *An Introduction to Astronomy.*

Moulton analyzes the special case of two stars each equal to the Sun in mass which, at a great distance apart, have a relative velocity that is zero. Let them approach each other, but assume that at the point of nearest approach they are as far apart as the Earth is from the Sun; their relative velocity at that instant will be about thirty-seven miles per second. Now, if they encounter no resistance to their motion, they will simply swing round each other and then separate again, moving along parabolic paths. If, however, one or both encounter resistance "from outlying nebulous or planetesimal matter, or from collision with a planet" their velocities will be reduced and their orbits may be transformed into elongated ellipses. Moulton states that, in the assumed case, if the resistance reduces the velocity by 1/200 of its amount, or 0.185 miles per second, the stars will, after their nearest approach, recede to a distance of only 100 astronomical units, and that the reduction of velocity will generate only as much heat as the Sun radiates in about eight years, not enough to affect the stars seriously. The eccentricity of their orbits will be about 0.98, their revolution period about 250 years. Collision with a planet comparable to *Jupiter* in mass would suffice to bring about the results described.

If no subsequent collisions occur, the two stars will continue to move in very elongated ellipses about their common center of gravity. "If there are subsequent collisions with other planets or with any other material in the vicinity of the stars, their points of nearest approach will not be appreciably changed unless the collisions are far from the perihelion point, their points of most remote recession will be diminished by each collision and the result is that both the period and the eccentricity of the orbit will be decreased as long as the process continues. If this is the correct theory of the origin of binary stars, those whose periods and eccentricities are small, are older on the average, at least as binary stars, than those whose periods and eccentricities are large . . . "

It will be noted that the conditions under which two stars are imagined to be converted into a binary system are much more plausible in Moulton's development of the collision principle than in Stoney's, and that he introduces variables enough to make the theory competent to account for any particular form of binary star orbit actually observed; we may vary the masses, the initial relative velocity, the degree of approach, the amount of resistance encountered. Moulton does not mention triple or quadruple systems in this connection; but to account for these it would seem necessary simply to imagine the encounter, under suitable conditions, of binaries relatively old with other single or binary stars.

Objections from a philosophical point of view might easily be urged against this theory, but it will suffice to present a very serious one raised by the facts of observation, an objection frankly acknowledged by Moulton and recognized also by Stoney—the very great number of the binary systems and the extreme rarity of near approaches or partial collisions of stars. Writing before the modern era of double star discovery had opened and when spectroscopic binaries were entirely unknown, Stoney says that if his theory is correct "we must conclude the sky to be peopled with countless hosts of dark bodies so numerous that those which have met with such collisions as to render them now visibly incandescent must be in comparison few indeed." Moulton, after pointing out the difficulty, says that its seriousness "depends upon the length of time the stars endure, about which nothing certain is known." That is quite correct, but it is none the less possible as a matter of statistical calculation to estimate the probable frequency of stellar encounters, and thus to apply a numerical test to the probability of the theory. Such a calculation has recently been made (for a different purpose) by Professor J. H. Jeans.[1]

Assume 1,000 million stars (perhaps ten times as many as are visible with our greatest telescopes) to be distributed in a space within a parallactic distance of 0.001″ from the Sun, a space which corresponds to a sphere having a radius of 3,260 light-years. Their mean distance apart, if they are arranged in cubical piling, is then $10^{18.7}$ centimeters, or about 330,000 times the Earth's distance from the Sun. Under reasonable assumptions as to masses and velocities, Jeans finds that "a star is only likely to experience a non-transitory encounter about once in 4×10^{13} (40 million million) years." To put it differently, only one star in every 4,000 will experience a non-transitory encounter in 10,000 million years. By a non-transitory encounter is meant one which will produce serious tidal deformations in one of the stars. If the stars are taken equal to the Sun in mass and given a relative velocity of forty kilometers per second, their separation at point of nearest approach must be not greater than the radius of *Jupiter's* orbit (about five astronomical units) to produce a lasting or non-transitory encounter. Transitory encounters, encounters in which the distance at point of nearest approach exceeds five astronomical units, will naturally be more frequent, but, with the data assumed, in 10,000 million years only about one star in three will have another star approach within 200 astronomical units of its center; that is, within seven times the distance of Neptune from the Sun!

It is perhaps conceivable that a single encounter similar to the one Moulton describes might occur, but it is quite inconceivable that more than a very few should occur in any length of time that may reasonably be assigned for the age of our present stellar universe. Still more inconceivable is the sup-

[1] The Motion of Tidally-distorted Masses, with Special Reference to Theories of Cosmogony.—*Mem. R. A. S.*, *72*, Part I, 1917.

position that a given star should have two or more encounters such as might produce triple or multiple systems. When we add the fact that the presence of a resisting medium adequate to convert hyperbolic or parabolic, into elliptic motion is a necessary condition that such encounters may result in binary systems, it appears that the theory would hardly be tenable even if we had but a few hundred systems to account for instead of the tens of thousands actually known.

The two theories which remain to be considered both assume that the binary stars originated in nebulae, and in this respect they are in harmony with practically every theory of stellar evolution from the time of Kant, Herschel and Laplace to the present day. The early theories were based upon nebulae more or less spheroidal in form, like the one assumed by Laplace in his famous hypothesis for the origin of our solar system. The contradictions in this hypothesis to fundamental principles of mechanics and the discovery that the majority of the nebulae are spiral in form led most astronomers, some twenty years ago, to favor the spirals as the antecedents to stellar systems. Now, the trend of opinion is towards the theory that the spirals are independent or 'island' universes, and that the irregular gaseous nebulae, like the Great Nebula in *Orion*, are the most primitive forms of matter known to us.

If a stellar system is to originate from a gaseous nebula,[2] it is clear that whatever initial form we assign to the nebular mass, and whatever other qualities we assign to the matter composing it, we must conceive it to be endowed with gravitational power which, sooner or later, will produce motion in its particles. The ensuing evolutionary process is thus sketched by Campbell:

"It will happen that there are regions of greater density, or nuclei, here and there throughout the structure which will act as centers of condensation, drawing surrounding materials into combination with them. The processes of growth from

[2] The spectroscope shows that the spirals are, as a rule, not gaseous nebulae; instead of the bright-line spectrum of luminous gases, they show a continuous spectrum, with dark absorption lines. Only four spirals are known, which, in addition to the continuous spectrum, show a few bright nebular lines.

nuclei originally small to volumes and masses ultimately stupendous must be slow at first, relatively more rapid after the masses have grown to moderate dimensions and the supplies of outlying materials are still plentiful, and again slow after the supplies shall have been largely exhausted. By virtue of motions prevailing within the original nebular structure, or because of inrushing materials which strike the central masses, not centrally but obliquely, low rotations of the condensed nebulous masses will occur. Stupendous quantities of heat will be generated in the building-up process. This heat will radiate rapidly into space because the gaseous masses are highly rarified and their radiating surfaces are large in proportion to the masses. With loss of heat the nebulous masses will contract in volume and gradually assume forms more and more spherical. When the forms become approximately spherical, the first stage of stellar life may be said to have been reached." [3]

If we start with the assumption that the binary systems as well as the single stars have developed from nebulae by some such process as Campbell has outlined, the question we have to consider is whether they were formed directly by condensation about separate nuclei, or whether, in a very early stage, they were single stars or spheroidal nebulae, dividing later under the stress of such internal forces as gravitation, radiation pressure and the forces of rotation, or under the strain of some external disrupting force. Each view has its advocates and its opponents, and so far as strict mathematical analysis goes, no definite answer has as yet been made.

The behavior of a rotating homogeneous incompressible fluid mass in equilibrium and free from external disturbance has been made the subject of a series of brilliant researches by such mathematicians as Maclaurin, Jacobi, Poincaré, and G. H. Darwin. It has been possible, under certain assumptions, to follow the transformations of form as the mass contracts under its own gravitation and heat radiation and to show that it passes from the initial sphere through a succession of spheroids,

[3] W. W. Campbell, 'The Evolution of the Stars and the Formation of the Earth'. *Scientific Monthly*, September–December, 1915.

ellipsoids and pear-shaped figures till a stage is reached where it seems certain that the next transformation will be a rupture into two masses. The analysis becomes too complicated to permit this step to be demonstrated mathematically.

The stars and the antecedent nebulae are not homogeneous nor incompressible, but it has been argued, first, I believe, by Dr. T. J. J. See, later by Darwin and others, that a nebula might none the less pass through a series of similar changes and ultimately form a double star. Darwin has shown that the two portions must have fairly comparable masses if the system, immediately after the rupture, is to be a stable one. Once formed, with components revolving in surface contact and in orbits practically circular, the agency of tidal forces is invoked to produce increase in the ellipticity of the orbit and in the length of the major axis.

The potency of tidal friction, within limits to be noted later, is undeniable, and the whole theory is made exceedingly plausible by its apparent ability to explain many of the facts of observation. Thus, according to the classical theory of stellar evolution, the stars of spectral Classes Oe and B are the 'young' stars; those of Classes A, F, G, and K, progressively older, the red stars of Class M, the 'old' stars, age being measured not in duration of time but by the stage of development reached. Now, as we have seen, the spectroscopic binaries which, on the average, have the shortest periods and orbits most nearly circular, are young stars; spectroscopic binaries of long period and high eccentricity belong, on the average, to the later spectral classes; they are old stars. Visual binaries of known period are wanting among stars of very early or very late spectral class. Campbell, who favors the fission theory, argues that the binaries of very early type have components too close together to be separately visible, while those of very late spectral class have their components so far apart that the revolution periods are exceedingly long. He also thinks that, in many red binary pairs, the fainter component may have become so faint as to be no longer visible.

Russell finds further support for the fission theory in the numerous triple and quadruple systems. His analysis leads him to conclude that, with a distribution of masses such as is actually found in double star systems, if further division occurs, the resulting multiple system must consist of a (relatively) wide pair one or both of whose components are themselves more closely double, the distance separating the components of the closer pair or pairs being "less than about one-fifth that of the wider pair—usually much less." He finds support for his conclusions in his discussion of the distance ratios in the known triple and quadruple systems.

The alternative theory, that double stars had their origin in separate nuclei in the parent nebula, was first suggested a century ago by Laplace. In Note VII to his *Système du Monde* he remarks:

"Such groups (as the *Pleiades*) are a necessary result of the condensation of nebulae with several nuclei, for it is plain that the nebulous matter being constantly attracted by these different nuclei must finally form a group of stars like the *Pleiades*. The condensation of nebulae with two nuclei will form stars in very close proximity, which will turn one around the other, similar to those double stars whose relative motions have already been determined." [4]

Laplace's theory is quite generally accepted for those wide and irregular groups, like θ *Orionis*, "for which the fission theory gives no explanation," and it is not without its adherents even for the usual type of binary system. The modern writer who has adopted it most explicitly is the very man who first formulated the fission theory, Dr. See. His discussion of the binary stars in the second volume of his *Researches on the Evolution of the Stellar Systems* is devoted more particularly to the development of the systems after their initial formation but on page 232 we find the statement, "It is evident . . . that the resulting mass-ratio in a system depends on the supply of nebulosity and the original nuclei already begun and slowly developing in the nebula while it was still of vast extent and great tenuity;" and, on page 584, the even more definite statement, "When a double star had been formed *in the usual*

[4] See *Essays in Astronomy*, p. 501. (Edited by E. S. Holden; D. Appleton & Co., 1900).

way (Italics mine) by the growth of separate centers in a widely diffused nebula . . . "

Moulton, as we have seen, favors the theory that binaries arose from entangling encounters of independent stars, but as between the two theories now under discussion, remarks " . . . we are led to believe that if binaries and multiple stars of several members have developed from nebulas, the nebulas must originally have had well-defined nuclei. The photographs of many nebulas support this conclusion." And Russell, while advocating the fission theory, admits that "The close pairs, almost in contact, revealed to us among the variable stars may be accounted for on either theory. The apparently universal fact that the components of a binary are comparable in mass is what might be expected as a consequence of the fission theory, but would probably have to be a postulate of the other."

Now it is obvious that the separate-nuclei theory affords sufficient latitude for the explanation of any conceivable system. We may imagine the two nuclei to be so remote from each other originally that their initial approach will be practically along the paths of parabolas and then invoke the action of a resisting medium to convert the motion first into that in an elongated ellipse and later into ellipses of ever smaller eccentricity and major axes; or we may imagine them placed so near each other that their relative motion is at first in small circular orbits which are afterward enlarged and made more eccentric by the action of tidal forces; and, in fact, this, crudely put, is the argument which See makes. That is, he considers that both of these opposing forces are actually effective in producing the binary systems as we now see them, one or the other becoming dominant in a particular system.

The fundamental objection to the separate nuclei theory is that we really do not explain anything; to use Moulton's words, "we only push by an assumption the problem of explaining the binary systems a little farther back into the unknown."

Russell's specific objection, based upon his study of the triple and quadruple systems is also of great force. Why should these systems almost invariably consist of a comparatively close binary pair attended by a third star or by another close binary pair at a distance relatively great? As Russell says, "Not only is there no apparent reason for it, but if we try to retrace in imagination the history of such a system, through stages of greater and greater diffusion as we penetrate farther into the past (keeping in mind that the moment of momentum of the whole system must remain constant), it is hard to form any idea of the history of the nuclei which will finally form a close and rapidly revolving pair, attended by a distant companion."

All things considered, the theory which has most in its favor is the fission theory, though it must be admitted that there are very serious difficulties about accepting it unreservedly. These relate, first of all, to the possibility of the initial division of the parent nebula. I say nebula rather than star because Moulton's researches make it certain that binary stars in which the masses of the components are comparable and the periods such as we find in the visual systems must at the time of separation have had densities extremely small. This, in Moulton's words, "removes the chief support of the belief that there is any such thing as fission among the stars simply because of rapid rotation." Jeans comes to the same conclusion as to the potency of rotation, but he finds that "gravitation also will tend toward separation," and that "a nebula can split into two parts under gravitation alone, the two nuclei being held apart by the pressure of the layer of gas which separates them, instead of by the so-called centrifugal force."

But, granting the formation of a system by fission, and granting also its stability, which, as Russell says "may well be accepted, on . . . physical grounds, unless direct mathematical evidence is produced to the contrary," we have still to ask how such a series of orbits as we actually observe could develop. Tidal friction is quite inadequate, as Moulton and also Russell have shown. At most, if the masses are equal, this force could increase the initial period to only about twice its value; if unequal, but comparable (and we must remember that the greatest inequality of mass yet observed in a binary star is only about 1:6), it might lengthen the period several fold, but certainly not from a few hours or days to many years, no matter how long the time. Similarly, while tidal friction can increase the eccentricity up to a certain point, it cannot transform a circular orbit into the highly eccentric orbits in which many of the visual, and even some of the spectroscopic binaries revolve. Campbell meets this difficulty with the statement, "if the tidal force is not competent to account for the observed facts . . . , some other separating force or forces must be found to supply the deficiency." No one can say that such forces do not exist; we can only say that as yet they remain unknown.

The general conclusion of our discussion is that no theory of the origin and development of binary systems so far formulated seems to satisfy fully the facts of observation enumerated at the beginning of the chapter. This need occasion no surprise, for binary star astronomy is still in its infancy. The entire history of the spectroscopic binary stars extends over less than thirty years, and these three decades have witnessed also some of the most important developments in the history of the visual binaries. Further, the researches which provide the mathematical bases upon which our theories must be built date for the most part from very recent years.

We must remember also that the present is an age of disturbance and upheaval in nearly every branch of natural science. Physicists and chemists have been submitting the fundamental principles of their sciences to searching criticism with results that are well-nigh revolutionary; biologists of the highest ability are questioning some of the basic doctrines of the origin of species. It would be strange indeed if at this time it were possible to formulate a theory of the origin of the binary stars, or a general theory of stellar evolution, which would be really satisfactory, or even sufficiently satisfactory to meet with general acceptance.

Thirty years ago, and even more recently, few seriously questioned the general outlines of the classical, or as one astronomer terms it, the conventional theory of stellar evolution. Stars were assumed to originate from nebulae somewhat after the manner described by Campbell in the passage which I have quoted; then, as the result of continuous radiation of heat, with the consequent contraction in volume and increase of density, to pass progressively through the stages marked for us by spectra of Classes B, A, F, G, K, and M, from white-hot stars to cooler yellow stars, orange stars and finally to red stars and thence to extinction. Facts of observation developed in more recent years seem to run counter to this order of development and raise difficulties so serious that a radically different order has been proposed and has won many adherents. This is the so-called two-branched order of development which has been most completely formulated by Russell. According to it the very young stars are red and of relatively low temperature; as they contract, they generate heat faster than it is radiated into space and the temperature rises. Consequently the stars pass through the spectral classes in the reverse order from that just given, and *if they are sufficiently massive* reach the white-hot state corresponding to a spectrum of Class B. From this point, with ever falling effective temperature, they pass through the spectral classes again in the normal order until they are once more red stars and finally become extinct. Less massive stars, the dwarfs, cannot reach the white-hot state, and turn at spectral Class A, or even F.

Campbell, who with the majority of astronomers, favors the classical theory of stellar evolution, finds in the observed facts relating to the spectroscopic and visual binaries strong support for his views; Russell and others see in them confirmation of their theory. Personally, I am inclined to prefer the classical theory of general stellar evolution and the fission theory of the origin of binary stars, *as working hypotheses*, frankly admitting, however, that the observed facts offer difficulties and objections which no means at present available can remove.

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See also the References at the end of Chapter VIII.

TABLE I[1]

List of Orbits of Visual Binary Stars

No.	Star	Mag.	Spec.	*1900.0*	*1900.0*	*P*	*T*
						y	
1	Σ 3062	6.5, 7.5	G5	$0^h\ 1.0^m$	+57° 53′	105.55	1836.07
2	Σ 2	6.8, 7.1	A3	3.8^m	+79° 10′	166.24	1890.87
3	Ho 212	5.6, 6.4	F	30.1^m	− 4° 9′	6.88	1905.27
4	β 395	6.4, 6.5	Ko	32.2^m	−25° 19′	25.0	1899.50
5	Σ 60	3.6, 7.9	F8	$0^h\ 43.0^m$	+57° 17′	507.60	1890.03
6	β 513	4.7, 7.2	A2	$1^h\ 53.7^m$	+70° 25′	52.95	1905.60
7	O Σ 38	5.4, 6.6	A	$1^h\ 57.8^m$	+41° 51′	55.0	1892.0
8	Σ 228	6.4, 7.3	Fo	$2^h\ 7.6^m$	+47° 1′	204.7	1891.59
9	β 524	5.6, 6.7	Fo	$2^h\ 47.4^m$	+37° 56′	33.33	1895.0
10	Σ 518	9.4, 10.8	Ao	$4^h\ 10.7^m$	− 7° 49′	180.03	1843.18
11	O Σ 79	7.5, 9.3	Go	14.2^m	+16° 17′	88.9	1897.8
12	β 883	7.9, 7.9	F5	$4^h\ 45.7^m$	+10° 54′	16.61	1907.03
13	Sirius	−1.6, 8.5	A	$6^h\ 40.8^m$	−16° 35′	49.32	1894.13
14	Castor	2.0, 2.8	A	$7^h\ 28.2^m$	+32° 6′	346.82	1969.82
15	Procyon	0.5, 13.5	F5	34.1^m	+ 5° 29′	39.0	1886.5
16	β 101	5.8, 6.4	Go	47.2^m	−13° 38′	23.34	1892.60
17	β 581	8.7, 8.7	G5	$7^h\ 58.8^m$	+12° 35′	46.5	1909.40
18	ζ Cancri	5.6, 6.3	Go	$8^h\ 1.5^m$	+17° 57′	60.083	1870.65
19	ε Hydrae	3.7, 5.2	F8	$8^h\ 41.5^m$	+ 6° 47′	15.3	1900.97
20	Σ 3121	7.6, 7.9	Ko	$9^h\ 12.0^m$	+29° 0′	34.0	1878.30
21	Σ 1356	5.9, 6.7	Go	23.1^m	+ 9° 30′	116.74	1840.82
22	O Σ 208	5.0, 5.6	A2	45.3^m	+54° 32′	99.70	1882.46
23	A. C. 5	5.8, 6.1	A2	$9^h\ 47.6^m$	− 7° 38′	72.76	1880.54
24	ξ Urs. Maj.	4.4, 4.9	Go	$11^h\ 12.8^m$	+32° 6′	59.81	1875.76
25	O Σ 234	7.3, 7.7	F5	25.4^m	+41° 50′	77.0	1880.10
26	O Σ 235	5.8, 7.1	F5	$11^h\ 26.7^m$	+61° 38′	71.9	1909.0
27	γ Virg.	3.6, 3.7	F	$12^h\ 36.6^m$	− 0° 54′	182.30	1836.42
28	42 Com. Ber.	5.2, 5.2	F5	$13^h\ 5.1^m$	+18° 3′	25.335	1885.54
29	O Σ 269	7.2, 7.7	A5	28.3^m	+35° 25′	48.8	1882.80
30	β 612	6.3, 6.3	F2	$13^h\ 34.6^m$	+11° 15′	23.05	1907.22
31	β 1111	7.4, 7.4	Ao	$14^h\ 18.5^m$	+ 8° 54′	44.32	1920.4
32	α Centauri	0.3, 1.7	G5 K5	32.8^m	−60° 25′	78.83	1875.68
33	Σ 1865	4.4, 4.8	A2	36.4^m	+14° 9′	130.	1898.0
34	Σ 1879	7.8, 8.8	F8	41.4^m	+10° 5′	238.0	1868.30
35	O Σ 285	7.5, 8.0	F5	41.7^m	+42° 48′	97.93	1883.56

[1]The orbits listed in this Table are discussed in Chapter VIII.

TABLE I

List of Orbits of Visual Binary Stars

e	a	Ω	i	ω	*Angle*	*Computer*	*No.*
0.466	1.44″	37.4°	±46.1°	98.7°	Inc.	Doberck	1
0.40	0.55″	154.9°	70.2°	316.1°	Dec.	Glasenapp	2
0.725	0.242″	38.7°	53.45°	66.8°	Inc.	Aitken	3
0.171	0.66″	112.8°	76.0°	152.7°	Inc.	Aitken	4
0.522	12.21″	99.2°	31.6°	88.9°	Inc.	Doberck	5
0.347	0.61″	81.4°	35.9°	6.2°	Inc.	See	6
0.82	0.346″	113.5°	76.6°	201.2°	Dec.	Hussey	7
0.41	0.97″	102.6°	56.45°	290.7°	Inc.	Rabe	8
0.60	0.16″	127.1°	33.5°	325.0°	Dec.	Aitken	9
0.134	4.79″	150.8°	63.25°	319.55°	Dec.	Doolittle	10
0.625	0.57″	66.0°	56.2°	129.8°	Inc.	Aitken	11
0.445	0.19″	34.2°	9.35°	190.0°	Inc.	Aitken	12
0.590	7.55″	43.27°	+44.55°	213.7°	Dec.	Lohse	13
0.44	5.756″	33.9°	63.6°	277.6°	Dec.	Doberck	14
0.324	4.05″	150.7°	14.2°	36.8°	Inc.	L. Boss	15
0.75	0.69″	99.7°	79.8°	74.65°	Inc.	Aitken	16
0.40	0.53″	116.5°	59.4°	282.0°	Inc.	Aitken	17
0.339	0.856″	Indet.	0.0°	183.55°	Dec.	Doberck	18
0.65	0.23″	104.4°	+49.95°	270.0°	Inc.	Aitken	19
0.33	0.67″	28.25°	75.0°	127.5°	Inc.	See	20
0.56	0.844″	144.3°	66.2°	122.1°	Inc.	Doberck	21
0.44	0.32″	186.5°	14.6°	342.2°	Inc.	Doberck	22
0.60	0.41″	197.95°	37.1°	46.9°	Dec.	Schroeter	23
0.411	2.513″	100.7°	53.4°	129.2°	Dec.	Nörlund	24
0.302	0.35″	157.5°	50.8°	206.6°	Inc.	See	25
0.40	0.78″	78.5°	43.6°	135.0°	Inc.	Aitken	26
0.887	3.74″	40.4°	29.9°	260.4°	Dec.	Doberck	27
0.496	0.674″	11.2°	90.0°	278.7°		Doberck	28
0.36	0.325″	46.2°	71.3°	32.6°	Inc.	See	29
0.52	0.225″	33.85°	50.4°	357.95°	Inc.	Aitken	30
0.15	0.26″	42.9°	46.4°	146.3°	Inc.	Aitken	31
0.512	17.65″	25.05°	+79.04°	52.35°	Inc.	Lohse	32
0.96	0.62″	129. °	39.7°	129. °	Dec.	Hertzsprung	33
0.70	1.06″	74.1°	57.6°	208.6°	Dec.	v. Biesbroeck	34
0.595	0.34″	Indet.	0.0°	262.85°	Dec.	v. Biesbroeck	35

TABLE I—(*Continued*)

No.	Star	Mag.	Spec.	1900.0	1900.0	P	T
						y	
36	Σ 1888	4.8, 6.7	G5	$14^h 46.8^m$	+19° 31′	159.54	1909.22
37	η Cor. Bor.	5.6, 6.1	Go	$15^h 19.1^m$	+30° 39′	41.56	1892.26
38	Σ 1938	7.2, 7.8	Ko	20.7^m	+37° 42′	244.37	1864.95
39	O Σ 298	7.4, 7.7	Ko	32.5^m	+40° 8′	56.653	1882.86
40	γ Cor. Bor.	4.0, 7.0	Ao	38.6^m	+26° 37′	87.8	1841.5
41	ξ Scorpii	4.8, 5.1	F8	$15^h 58.9^m$	−11° 6′	44.70	1905.39
42	Σ 2026	9.0, 9.5	K5	$16^h 11.1^m$	+ 7° 37′	242.10	1907.64
43	Σ 2055	4.0, 6.1	Ao	25.9^m	+ 2° 12′	134.	1811.5
44	ζ Herculis	3.0, 6.5	Go	37.5^m	+31° 47′	34.46	1898.77
45	Σ 2107	7.0, 8.5	F5	$16^h 47.9^m$	+28° 50′	221.95	1896.64
46	β 416	6.0, 8.5	K5	$17^h 12.2^m$	−34° 53′	41.47	1891.45
47	Σ 2173	5.9, 6.2	G	25.2^m	− 0° 59′	46.0	1915.2
48	μ Herculis	10.0, 10.5	Mb	42.6^m	+27° 47′	43.23	1880.20
49	τ Ophiuchi	5.3, 6.0	F	$17^h 57.6^m$	− 8° 11′	223.82	1814.79
50	70 Ophiuchi	4.1, 6.1	Ko	$18^h 0.4^m$	+ 2° 31′	87.858	1895.90
51	99 Herculis	5.2, 10.5	F8	3.2^m	+30° 33′	53.51	1887.84
52	A 88	7.2, 7.2	F8	33.2^m	− 3° 17′	12.12	1910.10
53	β 648	5.2, 8.7	Go	53.3^m	+32° 46′	45.85	1914.15
54	ζ Sagittarii	3.4, 3.6	A2	56.2^m	−30° 1′	21.17	1900.37
55	γ Cor. Austr.	5.0, 5.0	F8	$18^h 59.7^m$	−37° 12′	124.65	1878.46
56	Secchi 2	8.7, 8.7	G5	$19^h 7.8^m$	+38° 37′	58.	1894.0
57	ζ Sagittae	5.4, 6.4	A2	44.5^m	+18° 53′	25.20	1914.11
58	O Σ 400	7.5, 8.5	G5	$20^h 6.9^m$	+43° 39′	81.04	1888.23
59	β Delph.	4.0, 5.0	F5	32.9^m	+14° 15′	26.79	1883.04
60	Σ 2729	6.3, 7.6	F	46.1^m	− 6° 0′	135.6	1899.8
61	ε Equulei	5.8, 6.3	F5	$20^h 54.1^m$	+ 3° 55′	97.4	1873.5Ap
62	δ Equulei	5.3, 5.4	F5	$21^h 9.6^m$	+ 9° 36′	5.70	1901.35
63	τ Cygni	3.8, 8.0	Fo	10.8^m	+37° 37′	47.0	1889.60
64	κ Pegasi	5.0, 5.1	F5	$21^h 40.1^m$	+25° 11′	11.35	1897.8
65	Krueger 60	9.3, 10.8	Ma	$22^h 24.5^m$	+57° 12′	54.9	1929.3
66	β 80	8.3, 9.3	G	$23^h 13.8^m$	+ 4° 52′	95.2	1905.0
67	β 1266	8.3, 8.4	F5	25.5^m	+30° 17′	36.0	1911.35
68	85 Pegasi	5.8, 11.0	Go	$23^h 56.9^m$	+26° 33′	26.3	1883.5
1	Σ 73	6.1, 6.7	Ko	$0^h 49.6^m$	+23° 5′	109.07	1930.39
2	Σ 186	7.0, 7.0	Go	$1^h 50.7^m$	+ 1° 21′	136.	1894.0
3	Σ 483	8.0, 9.5	G5	$3^h 57.4^m$	+39° 14′	135.5	1907.75
4	O Σ 82	7.9, 9.5	Go	$4^h 17.1^m$	+14° 49′	97.94	1835.03
5	β 552	7.0, 10.0	F5	$4^h 46.2^m$	+13° 29′	56.0	1887.0

e	a	Ω	i	ω	*Angle*	*Computer*	*No.*
0.514	4.97″	170.8°	39.3°	337.0°	Dec.	Lohse	36
0.272	0.89″	25.25°	58.5°	218.0°	Inc.	Lohse	37
0.568	1.44″	169.4°	44.8°	26.1°	Dec.	Lohse	38
0.584	0.88″	2.1°	65.85°	21.9°	Inc.	Celoria	39
0.42	0.73″	111.0°	84.2°	99.2°	Dec.	Lewis	40
0.75	0.72″	27.2°	29.1°	343.6°	Inc.	Aitken	41
0.722	1.78″	10.2°	45.7°	162.2°	Dec.	Matzdorff	42
0.68	0.99″	87.7°	30.3°	123.5°	Inc.	Lewis	43
0.458	1.35″	51.6°	47.5°	113.3°	Dec.	Comstock	44
0.522	0.85″	179.6°	23.35°	123.5°	Inc.	Rabe	45
0.552	1.86″	131.0°	49.0°	64.0°	Dec.	Voûte	46
0.18	1.06″	153.7°	80.75°	322.2°	Dec.	Aitken	47
0.20	1.30″	60.8°	63.15°	182.0°	Inc.	Aitken	48
0.534	1.307″	76.2°	66.1°	17.75°	Inc.	Doberck	49
0.499	4.56″	122.96°	58.57°	193.64°	Dec.	Lohse	50
0.763	1.11″	75.0°	38.3°	93.7°	Inc.	Lohse	51
0.273	0.176″	2.4°	62.4°	270.0°	Dec.	Aitken	52
0.305	1.04″	52.5°	62.35°	335.7°	Dec.	Aitken	53
0.185	0.565″	75.5°	69.4°	1.4°	Dec.	Aitken	54
0.332	2.14″	53.5°	148.1°	169.55°	Dec.	Doberck	55
0.50	0.40″	90. °	68. °	0. °	Dec.	Russell	56
0.85	0.32″	4.6°	78.1°	65.0°	Dec.	v. Biesbroeck	57
0.46	0.47″	157.1°	59.9°	7.0°	Dec.	Burnham	58
0.35	0.48″	178.55°	62.25°	351.2°	Inc.	Aitken	59
0.35	0.64″	164.8°	62.3°	73.3°	Inc.	Aitken	60
0.72	0.61″	106.8°	85.5°	0.0°	Dec.	Russell	61
0.39	0.27″	21.0°	81.0°	164.5°	Dec.	Aitken	62
0.22	0.91″	149.8°	42.7°	105.5°	Dec.	Aitken	63
0.49	0.29″	109.2°	77.5°	106.1°	Dec.	Lewis	64
0.182	2.86″	113.6°	39.0°	161.0°	Dec.	Russell	65
0.77	0.72″	6.2°	22.95°	98.0°	Inc.	Aitken	66
0.24	0.24″	59.1°	62.15°	163.0°	Dec.	Aitken	67
0.46	0.82″	115.63°	53.08°	266.12°	Inc.	Bowyer and Furner	68
0.77	0.94″	109.5°	39.2°	71.1°	Inc.	Rabe	1
0.67	1.15″	42.6°	73.9°	226.8°	Inc.	Lewis	2
0.79	1.77″	23.1°	68.0°	213.4°	Dec.	See	3
0.50	0.94″	39.8°	59.8°	68.1°	Dec.	Hussey	4
0.345	0.53″	4.1°	56.7°	90.2°	Inc.	See	5

TABLE I—(*Continued*)

No.	Star	Mag.	Spec.	1900.0	1900.0	P	T
						y	
6	β 794	7.0, 8.3	F8	$11^h\ 48.3^m$	+74° 19′	42.0	1914.25
7	Σ 2123	7.7, 7.7	F5	$12^h\ 1.0^m$	+69° 15′	103.3	1860.50
8	Σ 1639	6.7, 7.9	A5	19.4^m	+26° 8′	180.	1892.0
9	γ Centauri	3.2, 3.2	A	$12^h\ 36.0^m$	−48° 25′	211.9	1851.6
10	Σ 1768	5.0, 8.0	Fo	$13^h\ 33.0^m$	+36° 48′	220.0	1866.5
11	Σ 1785	7.6, 8.0	K2	44.5^m	+27° 29′	199.2	1913.04
12	β 1270	8.6, 8.7	F5	$13^h\ 58.8^m$	+ 8° 58′	32.5	1912.2
13	Σ 1909	5.3, 6.2	Go	$15^h\ 0.5^m$	+48° 3′	204.74	1793.48
14	γ Lupi	3.6, 3.8	B3	28.5^m	−40° 50′	83.0	1845.0
15	π² Urs. Min.	7.0, 8.0	F2	$15^h\ 45.1^m$	+80° 17′	115.	1902.7
16	Δ 15	8.4, 8.7	K5	$16^h\ 40.8^m$	+43° 40′	109.	1897.8
17	Σ 2438	6.8, 7.4	A2	$18^h\ 55.8^m$	+58° 5′	233.0	1882.50
18	Σ 2525	8.0, 8.2	F8	$19^h\ 22.5^m$	+27° 7′	243.9	1887.09
19	O Σ 387	7.2, 8.2	F5	$19^h\ 46.0^m$	+35° 4′	90.0	1838.0

e	*a*	Ω	*i*	ω	*Angle*	*Computer*	*No.*
0.50	0.345″	109.2°	52.75°	225.0°	Inc.	Aitken	6
0.49	0.32″	56.9°	49.7°	79.1°	Dec.	See	7
0.70	0.71″	109.2°	58.15	18.1°	Dec.	Lewis	8
0.30	1.92″	3.35°	81.8°	285.0°	Dec.	Doberck	9
0.87	1.12″	31.4°	36.6°	257.1°	Dec.	Lewis	10
0.40	2.55″	169.1°	38.9°	159.1°	Inc.	v. Biesbroeck	11
0.42	0.22″	156.6°	38.7 °	12.15°	Inc.	Aitken	12
0.445	3.58″	58.7°	83.1 °	25.0°	Inc.	Doberck	13
0.70	1.10″	93.5°	90. ±°	90. ±°		See	14
0.80	0.42″	16.3°	62.25°	165.0°	Dec.	Aitken	15
0.55	1.24″	157.0°	72.0°	201.0°	Dec.	Evans	16
0.916	0.53″	Indet.	0.0°	178.3°	Dec.	See	17
0.918	0.952″	9.0°	47.3°	274.4°	Dec.	Doberck	18
0.60	0.66″	129.55°	65.75°	284.7°	Dec.	Doberck	19

TABLE II[1]

List of Orbits of Spectroscopic Binary Stars

A star preceding the figures in the column T indicates that the time is measured from a pa
ticular phase of the curve; from the epoch of light maximum, as in most of the Cepheid variabl
from the epoch of light minimum as in ζ *Geminorum*, or from an epoch of maximum or of zero relati
radial velocity as in π^5 *Orionis*.

No.	*Star*	*Mag.*	*Sp.*	*1900.0*	*1900.0*	*P*	*T* 2410000+
1	α Androm.	2.15	A	$0^h\ 3.2^m$	$+28°\ 32'$	$96^d.67$	7882.40
2	Boss 82	5.16	A2	22.9^m	$+43°\ 51'$	3.956^-	8841.59
3	13 Ceti	5.24	F	30.1^m	$-\ 4°\ 9'$	2.082^-	7484.48
4	π Androm.	4.44	B3	31.5^m	$+33°\ 10'$	143.67^-	8564.14
5	π Cass.	5.02	A5	37.8^m	$+46°\ 29'$	1.964^+	9970.03
6	23 Cass.	5.39	B8	41.1^m	$+74°\ 18'$	33.75	10577.41
7	ζ Androm.	4.30	K	42.0^m	$+23°\ 43'$	17.767^+	10024.881
8	ν Androm.	4.42	B3	$0^h\ 44.3^m$	$+40°\ 32'$	4.283^-	
9	α Urs. Min.	Var.	F8	$1^h\ 22.6^m$	$+88°\ 46'$	3.968^+	4890.04
						11.9 yr.	4509.0
10	φ Persei	Var.	Bp	37.4^m	$+50°\ 11'$	126.5	8290.42
						63.25	*8326.58
11	α Triang.	3.58	F5	47.4^m	$+29°\ 6'$	1.737^-	10793.821
12	ω Cass.	5.03	B8	48.2^m	$+68°\ 12'$	69.92	10426.02
13	β Arietis	2.72	A5	$1^h\ 49.1^m$	$+20°\ 19'$	107.0	7632.0
14	RZ Cass.	Var.	A	$2^h\ 39.9^m$	$+69°\ 13'$	1.195^+	9449.732
15	π Arietis	5.30	B5	$2^h\ 43.7^m$	$+17°\ 3'$	3.854	10370.259
16	β Persei	Var.	B8	$3^h\ 1.7^m$	$+40°\ 34'$	2.867^+	*2.264
						1.899yr.	1901.855
17	ο Persei	3.94	B1	38.0^m	$+31°\ 58'$	4.419^+	8217.924
18	ξ Persei	4.05	Oe5	52.5^m	$+35°\ 30'$	6.951	8248.308
19	λ Tauri	Var.	B3	$3^h\ 55.1^m$	$+12°\ 12'$	3.953^-	7945.119
						34.60	7831.30
20	μ Persei	4.28	G	$4^h\ 7.6^m$	$+48°\ 9'$	284.	10061.97
21	b Persei	4.57	A2	10.7^m	$+50°\ 3'$	1.527^+	8956.166
22	υ_4 Eridani	3.59	B9	14.1^m	$-34°\ 2'$	5.010^+	7562.266
23	63 Tauri	5.68	A2	17.7^m	$+16°\ 32'$	8.425	9819.0
24	θ_2 Tauri	3.62	A5	22.9^m	$+15°\ 39'$	140.70	8054.723
25	d Tauri	Var.	A2	30.2^m	$+\ 9°\ 57'$	3.571^+	9734.992

[1] The orbits listed in this Table are discussed in Chapter VIII.

TABLE II

List of Orbits of Spectroscopic Binary Stars

When two sets of elements are given, the lower figures for K and for $a \sin i$ relate to the second component if the two values for ω differ by 180°; if the lower value of P is an aliquot part of the upper value, the lower line of figures relates to an assumed secondary oscillation; in other cases the lower line relates to a third body definitely known or suspected in the system.

ω	e	K km.	V_o km.	$a \sin i$ million km.	$\frac{m_1^3 \sin^3 i}{(m + m_1)^2}$	*Computer*	*No.*
76.21°	0.525	30.75	−11.53	34.790	0.180⊙	Baker	1
233.2°	0.152	41.7	+ 2.04	2.240		Miss Udick	2
223.1°	0.062	34.35	+10.5	0.981	0.0087	Fox	3
350.53°	0.573	47.66	+ 8.83	77.200	0.876	Jordan	4
45.1°	0.009	117.76	+ 8.92	3.180+		Harper	5
225.1°		117.40		3.170+			
269.71°	0.405	16.32	− 4.06	7.020	0.0121	Young	6
182.22°	0.037	25.69	−29.83	6.272		Cannon	7
	0.000	75.63	−23.91	4.454		Jordan	8
		104.0		6.125			
80.0°	0.13	3.04		0.164	0.00001	Miss Hobe	9
293.0°	0.35	2.98	−14.8	166.800	0.0098		
347.29°	0.428	26.90	+ 3.20	42.298	0.189	Cannon	10
257.14°	0.107	6.96					
135.56°	0.121	12.10	−12.65	0.287		Harper	11
49.97°	0.30	29.64	−24.82	27.190	0.164	Young	12
19.7°	0.88	32.6	− 0.6	22.880	0.042	Ludendorff	13
154.7°	0.052	69.30	−38.32	1.137	0.0412	Jordan	14
78.27°	0.042	24.77	+ 7.81	1.310	0.0061	Young	15
21.0°	0.060	41.3	+ 3.40	1.630	0.021	Curtiss	16
	0.000	9.4	+ 4.1	89.000	0.060		
	0.000	111.92	+18.46	6.801	0.754	Jordan	17
		160.0		9.717			
99.18°	0.034	7.87	+15.40	0.752		Cannon	18
77.5°	0.061	56.18	+12.95	3.050	0.073	Schlesinger	19
	0.000	10.4		4.950	0.004		
301.99°	0.062	20.50	+ 7.83	80.000		Cannon	20
151.75°	0.22	41.89	+23.09	0.838		Cannon	21
331.75°		152.5		3.048			
124.33°	0.014	63.76	+17.83	4.393		Paddock	22
304.33°		64.85		4.468			
190.7°	0.16	36.5	+36.4	4.170	0.041	Jantzen	23
54.16°	0.717	27.12	+42.59	37.471		Plaskett	24
	0.000	72.68	+29.23	3.570	0.142	Daniel	25

TABLE II—(*Continued*)

No.	*Star*	*Mag.*	*Sp.*	*1900.0*	*1900.0*	*P*	*T* 2410000+
26	SZ Tauri	Var.	F8	31.4^m	+18° 20′	$3^d.148^+$	10016.187
27	τ Tauri	4.33	B5	36.2^m	+22° 46′	1.505^-	7892.500
28	9 Camelop.	4.38	B	44.1^m	+66° 10′	7.996^-	6480.35
29	π4 Orionis	3.78	B3	45.9^m	+ 5° 26′	9.519^+	8279.64
30	π5 Orionis	3.87	B3	49.0^m	+ 2° 17′	3.700^+	*7921.64
31	7 Camelop.	4.44	A2	$4^h\ 49.3^m$	+53° 35′	3.885^-	8281.176
32	14 Aurigae	5.14	A2	$5^h\ 8.8^m$	+32° 35′	3.789^-	10802.715
33	α Aurigae	0.21	G	9.3^m	+45° 54′	104.022	4899.5
34	β Orionis	0.34	B8p	9.7^m	− 8° 19′	21.90	7968.80
35	η Orionis	3.44	B1	19.4^m	− 2° 29′	7.990^-	5720.821
36	ψ Orionis	5.79	A	21.6^m	+ 3° 0′	2.526^-	7916.36
37	χ Aurigae	4.88	B1	26.2^m	+32° 7′	655.16	10629.78
38	δ Orionis	Var.	B	26.9^m	− 0° 22′	5.732^+	*9806.383
39	VV Orionis	Var.	B2	28.5^m	− 1° 14′	1.485^+ 120.0	9836.021 9819.
40	ι Orionis	2.87	Oe5	30.5^m	− 5° 59′	29.136	7587.991 7577.354
41	ζ Tauri	3.00	B3	31.7^m	+21° 5′	138.0	5769.9
42	125 Tauri	5.00	B3	33.5^m	+25° 50′	27.864	10471.607
43	Boss 1399	5.00	B3	35.8^m	− 1° 11′	27.160	7961.465
44	136 Tauri	4.54	A	47.0^m	+27° 35′	5.969	9362.52
45	β Aurigae	Var.	Ap	52.2^m	+44° 56′	3.960^+	7100.732
46	40 Aurigae	5.31	A	$5^h\ 59.6^m$	+38° 29′	28.28	10468.197
47	ν Orionis	4.40	B2	$6^h\ 1.9^m$	+14° 47′	131.26	7975.16
48	Boss 1607	5.50	A	18.0^m	+56° 20′	9.944	9341.776
49	RT Aurigae	Var.	G	22.1^m	+30° 34′	3.728^+	*3.423
50	γ Gemin.	1.93	A	31.9^m	+16° 29′	2175.0	101.6
51	ζ Gemin.	Var.	G	$6^h\ 58.2^m$	+20° 43′	10.154	*1.313
52	29 Can. Maj.	4.90	Oe	$7^h\ 14.5^m$	−24° 23′	4.393^+	7240.248
53	R Can. Maj.	Var.	F	14.9^m	−16° 12′	1.136^+	7966.576
54	σ Puppis	3.27	K5	26.1^m	−43° 6′	258.0	10419.0
55	α_1 Gemin.	2.85	A	28.2^m	+32° 6′	2.928^+	6828.057
56	α_2 Gemin.	1.99	A			9.219^-	6746.385
57	σ Gemin.	4.26	K	$7^h\ 37.0^m$	+29° 7′	19.605	5824.019
58	ε Volantis	4.46	B5	$8^h\ 7.6^m$	−68° 19′	14.168^+	9453.562
59	*ε Hydrae	3.48	F8	$8^h\ 41.5^m$	+ 6° 47′	5588.0	5375.0
60	κ Cancri	5.14	B8	$9^h\ 2.4^m$	+11° 4′	6.393	6486.897

ω	e	K km.	Vo km.	$a \sin i$ million km.	$\frac{m_1^3 \sin^3 i}{(m+m_1)^2}$	*Computer*	No.
76.66°	0.24	10.94	− 3.15	0.460	0.0004	Haynes	26
242.88°	0.087	44.34	+13.55	0.914		Parker	27
90.0°	0.30	9.0	− 2.25			Lee	28
152.27°	0.027	25.93	+23.27	3.393		Baker	29
	0.000	57.88	+24.20	2.945		Lee	30
217.14°	0.013	35.15	− 8.93	1.877		Harper	31
19.70°	0.033	21.56	−10.74	1.123		Harper	32
117.3°	0.016+	25.76	+30.17	36.848		Reese	33
297.3°		32.45		46.430			
254.76°	0.296	3.77	+22.62	1.109	0.0001	Plaskett	34
42.3°	0.016	144.75	+35.5	15.901	2.51	Adams	35
184.71°	0.065	144.12	+12.02	4.995	0.780	Plaskett	36
135.52°	0.171	20.53	− 0.15	182.300	0.56	Young	37
359.33°	0.098+	100.96	+20.15	7.926	0.605	Curtiss	38
	0.000	132.37	+20.77	2.704	0.358	Daniel	39
40.0°	0.30	13.0		20.460			
112.374°	0.742−	113.68	+21.53	30.560		Plaskett	40
	0.000	8.38		3.358			
9.8°	0.180	14.95	+16.4	27.900	0.046	Adams	41
335.0°	0.55	25.5	+14.8	8.160		Cannon	42
87.02°	0.765	93.04	+26.12	22.380		Harper	43
191.44°	0.022	48.9	−17.1	4.011		Cannon	44
11.44°		71.0					
	0.000	108.96	−18.1	5.934		Baker	45
		111.04		6.047			
178.40°	0.556	51.38	+16.91	16.550		Young	46
−1.60°		62.51		20.140			
1.58°	0.599	34.09	+22.10	49.270		Harper	47
152.9°	0.081	67.19	−13.74	9.127		Harper	48
95.016°	0.368	17.96	+21.43	0.856+		Duncan	49
16.31°	0.298	6.12	−12.28	174.720		Harper	50
333.0°	0.22	13.2	+ 6.8	1.798	0.0023	Campbell	51
37.64°	0.156	218.44	−12.12		4.58	Harper	52
195.86°	0.138	28.64	−39.70	0.443	0.0027	Jordan	53
350.0°	0.2	18.0	+88.5	62.570		Lunt	54
102.52°	0.01	31.76	− 0.98	1.279		Curtis	55
265.35°	0.503	13.56	+ 6.20	1.485		Curtis	56
330.25°	0.022	34.21	+45.80	9.220		Harper	57
	0.000	66.67	+ 9.68	12.999	0.437	Sanford	58
90.0°	0.65	8.45	+36.78	493.000		Aitken	59
162.26°	0.149	67.8	+26.3	5.890		Ichinohe	60

TABLE II—(*Continued*)

No.	Star	Mag.	Sp.	1900.0	1900.0	P	T 2410000+
61	a Carinae	3.56	B3	8.4^m	$-58° 33'$	$6^d.744$	6533.81
62	Boss 2484	5.70	A	10.8^m	$+47° 14'$	15.986	9408.027
63	κ Velorum	2.63	B3	19.0^m	$-54° 35'$	116.65	6459.00
64	ο Leonis	3.76	F5p	$9^h\ 35.8^m$	$+10° 21'$	14.498	4656.477
65	30 H Urs. Maj.	4.92	A	$10^h\ 16.9^m$	$+66° 4'$	11.583^+	8468.212
66	ω Urs. Maj.	4.84	A	48.2^m	$+43° 43'$	15.840^+	7991.101
67	β Urs. Maj.	2.44	A	$10^h\ 55.8^m$	$+56° 55'$	0.312^+	10895.896
68	93 Leonis	4.54	F8	$11^h\ 42.8^m$	$+20° 46'$	71.70	8088.405
69	Boss 3182	5.12	A5	$12^h\ 7.5^m$	$+78° 10'$	1.271	*10685.265
70	η Virginis	4.00	A	14.8^m	$-0° 7'$	71.9	7643.50
71	d2 Virginis	5.24	A	40.6^m	$+8° 13'$	38.3	10573.455
72	ε Urs. Maj.	1.68	Ap	49.6^m	$+56° 30'$	4.15yrs.	7576.40
73	α_2 Can. Ven.	2.90	Ap	$12^h\ 51.4^m$	$+38° 52'$	5.50	*3.84
74	ζ^1 Urs. Maj.	2.40	Ap	$13^h\ 19.9^m$	$+55° 27'$	20.536^+	9477.744
75	α Virginis	1.21	B2	19.9^m	$-10° 38'$	4.014^+	7955.846
76	Boss 3511	4.96	F	30.3^m	$+37° 42'$	1.613^-	7018.145
77	ν Centauri	3.53	B2	43.5^m	$-41° 11'$	2.625^+	10301.39
78	h Centauri	4.76	B5	47.5^m	$-31° 26'$	6.927	8733.25
79	η Boötis	2.80	G	$13^h\ 49.9^m$	$+18° 54'$	497.1	8240.60
80	α Drac.	3.64	A	$14^h\ 1.7^m$	$+64° 51'$	51.38	7403.284
81	d Boötis	4.82	F5	5.8^m	$+25° 34'$	9.604^+	7679.523
82	A Boötis	4.83	G5	13.8^m	$+35° 58'$	211.95	10561.18
83	δ Librae	Var.	A	$14^h\ 55.6^m$	$-8° 7'$	2.327	*1.89
84	β Cor. Bor.	3.72	Fp	$15^h\ 23.7^m$	$+29° 27'$	{ 40.9	8739.29
						{ 490.8	8804.4
85	α Cor. Bor.	Var.	A	30.5^m	$+27° 3'$	17.36	7742.55
86	A_2 Serp.	5.37	B8	40.9^m	$-1° 30'$	38.95	9528.597
87	β Scorpii	2.90	B1	$15^h\ 59.6^m$	$-19° 32'$	6.828^+	9163.923
88	θ Drac.	4.11	F8	$16^h\ 0.0^m$	$+58° 50'$	3.071^-	5368.962
89	σ Scorpii	3.08	B1	15.1^m	$-25° 21'$	0.247^-	10671.800
90	β Herculis	2.81	K	25.9^m	$+21° 42'$	410.575	5500.374

ω	e	K km	Vo km.	a sin i million km.	$\frac{m_1^3 \sin^3 i}{(m + m_1)^2}$	Computer	No.
115.84°	0.18	21.5	+23.3	1.960		Curtis	61
{355.2°	0.504	63.34	−13.11	12.026 }		Harper	62
{175.2°		73.64		13.981 }			
96.23°	0.19	46.5	+21.9	73.200		Curtis	63
	< 0.02	{54.0	+27.07	10.775	0.238 }	Plummer	64
		{63.1		12.571	0.378 }		
171.9°	0.381	34.07	− 0.10	5.020		Schlesinger	65
11.95°	0.264	20.64	−18.45	4.336		Parker	66
315.4°	0.124	1.25	−10.93	2660 km.	0.000	Guthnick and Prager	67
270.81°	0.008	26.54	+ 0.17	26.170		Cannon	68
	0.000	63.2	+ 0.3	1.104		Lee	69
185.0°	0.40	27.6	+ 2.2	25.750	0.126	Harper	70
223.35°	0.072	40.99	− 8.89	21.530		Cannon	71
43.35°		80.0					
55.8°	0.31	3.5	−12.9	69.360	0.0058	Ludendorff	72
110.0°	0.3	21.5	+ 1.0	1.500		Belopolsky	73
{103.96°	0.535⁻	69.22	− 9.64	16.400 }		Hadley	74
{283.96°		68.83		16.400 }			
328.0°	0.10	126.1	+ 1.6	6.930 }		Baker	75
148.0°		207.8		11.400 }			
201.5°	0.067	10.25	+ 7.09	0.227⁻	0.0002	Harper	76
	0.000	20.63	+ 9.05	0.745	0.0024	Wilson	77
147.23°	0.23	21.4	+ 5.2	1.984	0.0065	Paddock	78
315.20°	0.236	8.69	− 0.23	57.735		Harper	79
19.07°	0.384	46.25	−17.03	30.173		Harper	80
{273.0°	0.169	68.40	+ 9.80	8.904 }		Harper	81
{93.0°		72.05		9.380 }			
223.42°	0.54	18.02	+25.62	44.000	0.076	Young	82
29.2°	0.054	76.5	−45.0	2.450		Schlesinger	83
240.0°	0.4	3.10	−21.28	1.600 }		Cannon	84
	0.000	2.4		}			
312.2°	0.387	34.93	+ 0.36	7.671	0.060	Jordan	85
208.46°	0.773	50.52	−11.63	17.170	0.133	Jordan	86
{20.09°	0.270	125.66	−11.0	11.360 }		Daniel and Schlesinger	87
{200.09°		197.0		17.800 }			
126.11°	0.014⁺	23.47	− 8.36	0.990	0.004	Curtis	88
110.0°	0.05	39.0	+ 2.0	0.129	0.0014	Selga	89
24.60°	0.550⁻	12.78	−25.52	60.280	0.052	Reese and Plummer	90

TABLE II—(*Continued*)

No.	*Star*	*Mag.*	*Sp.*	*1900.0*	*1900.0*	*P*	*T* 2410000+
91	ε Urs. Min.	4.40	G5	56.2^m	+82° 12′	$39^d.482$	8005.75
92	ε Herculis	3.92	A	$16^h\ 56.5^m$	+31° 4′	4.024^-	8086.253
93	u Herculis	Var.	B3	$17^h\ 13.6^m$	+33° 12′	2.051^+	8125.80
94	Boss 4423	4.61	F	21.3^m	− 5° 0′	26.274^+	8411.524
95	ξ Serp.	3.64	A5	31.9^m	−15° 20′	2.292^+	9209.618
96	ω Drac.	4.87	F5	37.5^m	+68° 48′	5.280^-	7385.493
97	X Sagittarii	Var.	F8	41.3^m	−27° 48′	7.012^-	6723.05
98	Y Ophiuchi	Var.	G	47.3^m	− 6° 7′	17.121^-	*2.415
99	W Sagittarii	Var.	F5	$17^h\ 58.6^m$	−29° 35′	7.595^-	*6.20
100	μ Sagittarii	4.01	B8p	$18^h\ 7.8^m$	−21° 5′	180.2	4968.4
101	Y Sagittarii	Var.	G	15.5^m	−18° 54′	5.773^+	*4.46
102	108 Herculis	5.54	A	17.1^m	+29° 48′	5.515^-	9551.742
103	χ Drac.	3.69	F8	22.9^m	+72° 41′	281.8	4864.3
104	RX Herculis	Var.	A	26.0^m	+12° 32′	1.779^-	*9658.588
105	ζ₁ Lyrae	4.29	F	41.3^m	+37° 30′	4.300^-	8109.722
106	β Lyrae	Var.	B2p	46.4^m	+33° 15′	12.919^+	*9.867
107	50 Drac.	5.37	A	49.6^m	+75° 19′	4.120	10293.93
108		5.51	B3	50.2^m	+36° 51′	88.112	9220.727
109	δ_1 Lyrae	4.56	G5	$18^h\ 50.5^m$	+22° 32′	245.3	9805.0
110	113 Herculis	5.10	B8	$19^h\ 2.3^m$	+10° 55′	1.302^+	*8157.502
111	18 Aquilae	Var.	A	14.4^m	+19° 26′	3.381^-	8428.183
112	U Sagittae	4.58	B8p	16.0^m	−16° 8′	137.939	9648.72
113	υ Sagittarii	6.03	A	19.8^m	+16° 45′	7.390	10943.233
	2 Sagittae						
114		Var.	F	22.3^m	+42° 36′	0.567^-	*0.508
115	RR Lyrae	5.17	B8	34.3^m	+ 5° 10′	1.950^+	10054.331
	σ Aquilae						
116	SU Cygni	Var.	F5	40.8^m	+29° 1′	3.844	*2.5
117	η Aquilae	Var.	G	$19^h\ 47.4^m$	+ 0° 45′	7.176	*6.210
118	θ Aquilae	3.37	A	$20^h\ 6.1^m$	− 1° 7′	17.124^+	8261.914
119	β Capricorni	3.25	Gp	15.4^m	−15° 6′	1375.3	6035.0
120	α Pavonis	2.12	B3	17.7^m	−57° 3′	11.753	6379.90

ω	e	K km.	Vo km.	a sin i million kn.	$\frac{m_1^3 \sin^3 i}{(m+m_1)^2}$	Computer	No.
359.46°	0.011+	31.954	−11.398	17.346		Plaskett	91
180.0°	0.023	70.39	−24.03	3.890		Baker	92
0.0°		112.1		6.200			
66.15°	0.053	99.50	−21.16	2.800		Baker	93
246.15°		253.0		7.120			
14.48°	0.491+	47.49	+ 0.44	14.950		Parker	94
194.48°		50.67					
	0.000	19.35	−42.77	0.610		Young	95
333.76°	0.011−	36.26	−13.68	2.632		Turner	96
93.65°	0.40	15.2	−13.50	1.334		Moore	97
201.7°	0.163	7.70	− 5.10	1.790		Miss Udick	98
70.0°	0.320	19.5	−28.6	1.930	0.005	Curtiss	99
74.7°	0.441	64.5	− 7.0	143.500		Ichinohe	100
32.0°	0.16	19.0	+ 4.0	1.485		Duncan	101
	0.000	70.1	−20.2	5.320		Daniel and	102
		101.7		7.710		Jenkins	
119.0°	0.423	17.95	+32.38	62.020		Wright	103
	0.000	106.0	−18.5	2.590		Shapley	104
	0.000	51.24	−25.97	3.030		Jordan	105
0.15°	0.018	184.40	−20.95	32.750		Curtiss	106
180.15°		75.0		13.300			
151.0°	0.024	75.77	−10.1	4.291		Harper	107
331.0°		83.26		4.716			
204.55°	0.28	33.68	−25.85	39.220	0.309	Jordan	108
169.5°	0.12	16.0	−23.2	53.580	0.102	Wilson	109
	0.000	27.59	−18.65	0.494	0.0028	Jordan	110
44.14°	0.035	66.45	−19.13	3.090		Miss Fowler	111
28.6°	0.087	48.15	+12.1	91.010	1.582	Wilson	112
332.6°	0.05	52.95	+11.0	5.370		Young	113
152.6°		73.8		7.490			
96.85°	0.271	22.2	−68.7	0.166+	0.00057	Kiess	114
	0.000	163.52	− 5.0	4.380		Jordan	115
		199.0		5.340			
(345.8°)	0.21±	25.±	−33.4±	1.350±	0.0058±	Maddrill	116
68.91°	0.489	20.59	−14.16	1.773	0.0043	Wright	117
14.9°	0.681	46.0	−30.5	7.930		Baker	118
194.9°		63.0		10.860			
124.0°	0.44	22.2	−18.8	377.000		Merrill	119
224.80°	0.01	7.25	+ 2.0	1.170		Curtis	120

TABLE II—(*Continued*)

No.	*Star*	*Mag.*	*Sp.*	*1900.0*	*1900.0*	*P*	*T* 2410000+
121	T Vulpec.	Var.	F	47.2^m	+27° 52′	4^d.436$^-$	*3.678
122	57 Cygni	4.68	B3	20^h 49.7^m	+44° 0′	2.855	8554.770
123	β Cephei	Var.	B1	21^h 27.4^m	+70° 7′	0.190$^+$	9638.812
124	ι Pegasi	3.96	F5	22^h 2.4^m	+24° 51′	10.213$^+$	4820.966
125	2 Lacertae	4.66	B5	16.9^m	+46° 2′	2.616$^+$	8193.30
126	δ Cephei	Var.	G	25.4^m	+57° 54′	5.366$^+$	7888.428
127	12 Lacertae	5.18	B2	37.0^m	+39° 43′	0.193$^+$	10761.149
128	η Pegasi	3.10	G	22^h 38.3^m	+29° 42′	818.0	5288.7
129	9 Androm.	5.90	A2	23^h 13.7^m	+41° 13′	3.219$^+$	11059.921
130	1 H Cass.	4.89	B3	25.4^m	+58° 0′	6.067	8223.762
131	λ Androm.	4.00	K	32.7^m	+45° 55′	20.546	6683.46
132	Boss 6142	6.05	Bp	23^h 50.5^m	+56° 53′	13.435	10800.634
1	α Persei	1.90	F5	3^h 17.2^m	+49° 30′	4.094$^-$	7955.14
2	α Orionis	0.92	Ma	5^h 49.8^m	+ 7° 23′	6.0 yrs.	6693.
3	β Can. Maj.	1.99	B1	6^h 18.3^m	−17° 54′	0.25	
4	ρ Leonis	3.85	Bp	10^h 27.5^m	+ 9° 49′	12.28	8749.603
5	α Scorpii	1.22	Map	16^h 23.3^m	−26° 13′	5.80yrs.	6673.582

ω	e	K km.	Vo km.	a sin i million km.	$\frac{m_1^3 \sin^3 i}{(m + m_1)^2}$	Computer	No.
104.03°	0.440	17.63	− 1.39	0.966	0.0018	Beal	121
{45.0°	0.137	110.4	−16.2	4.200}		Baker	122
225.0°}		118.8		4.620}			
2.63°	0.040	15.798	−14.13	0.041⁻		Crump	123
251.81°	0.0085	47.99	− 4.12	6.740		Curtis	124
{180.0°	0.015	80.3	− 9.0	2.890}		Baker	125
0.0°}		98.8		3.550}			
85.385°	0.484	19.675	−16.83	1.271	0.0028	Moore	126
	0.000	16.92	−13.75	0.045	0.0001	Young	127
5.605°	0.155	14.20	+ 4.31	157.800		Crawford	128
40.57°	0.0365	73.56	− 4.87	3.240	0.133	Young	129
3.35°	0.224	59.06	−14.78	4.920		Baker	130
301.0°	0.086	7.07	+ 7.43	1.990		Burns	131
{339.56°	0.105	115.5	−26.7	21.200}		Young	132
159.56°}		167.		30.700}			
221.0°	0.47	0.93	− 3.43	0.046	0.0000	Hnatek	1
255.0°	0.24	2.45	+21.3	70.000	0.003	Bottlinger	2
	0.1 ±	9.8 ±	+33.7 ±	0.034⁻		Albrecht	3
180. ±°	0.5 ±	10. ±	+41.1			Schlesinger	4
289.0°	0.20	2.12	− 3.09			Halm	5

STAR LIST[1]

[1] No reference is made in this list to the Tables of Orbits on pages 290–305.

INDEX OF NAMES[1]

[1] No reference is made in this list to the Tables of Orbits on pages 290–305.

GENERAL INDEX